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THE DAMPING OF PENDULAR OSCILLATIONS USING A PARAMETRIC CONTROLLER

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Abstract. In the paper, the efficiency of a parametric damping system for the oscillations of a linear, initially, lightly damped pendulum has been studied. A closed-loop stabilization system included the feedback based on the deviation of the simple pendulum's position from the prearranged one. Our system was equipped with a controller whose structure followed from the Mathieu equation, and the variable coefficients were selected taking into account the proposed methodology. The parametric controller (PC) was shown to be able to shorten the settling time via increasing the damping. Because of this, it makes sense to add the PC to a system with a traditional offset-based P-controller to improve dynamic performance and to increase accuracy in the steady state. Moreover, the PC successfully suppressed harmonic external disturbances. The PC based on the offset of one link was constructed for a planar two-link pendulum. The controller damped lower frequency oscillations, but this turned out to be enough to significantly improve the stabilization response.

Keywords: linear lightly damped pendulum, first parametric resonance, feedback system, parametric controller

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ПОДАВЛЕНИЕ КОЛЕБАНИЙ МАЯТНИКА С ПОМОЩЬЮ ПАРАМЕТРИЧЕСКОГО РЕГУЛЯТОРА

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Аннотация. В работе исследована эффективность системы параметрического демпфирования колебаний линейного, изначально слабодемпфированного маятника. Для этого построена система стабилизации с обратной связью по отклонению положения однозвенного маятника от заданного. В замкнутой системе использован регулятор, структура которого следует из уравнения Матье, а варьируемые коэффициенты подобраны с учетом предложенной методики. Установлено, что параметрический регулятор (ПР) способен сократить время установления благодаря увеличению демпфирования. Поэтому ПР целесообразно добавлять в систему с традиционным пропорциональным регулятором

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по отклонению для улучшения динамических показателей и повышения точности в установившемся режиме. Показано, что ПР успешно подавляет гармонические внешние возмущения. Для плоского двухзвенного маятника построен ПР по отклонению одного звена. Регулятор демпфирует колебания с низшей частотой, но этого оказывается достаточно для существенного повышения быстродействия при стабилизации.

Ключевые слова: линейный слабодемпфированный маятник, первый параметрический резонанс, замкнутая система, параметрический регулятор, метод стационаризации

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Introduction

Parametric excitation can destabilize a system if parametric resonance occurs [1], or it can stabilize an unstable system [2]. Studies of oscillatory processes in periodically nonstationary dynamical systems focus primarily on constructing and refining the resonance regions in the plane of the parameters of the periodic coefficient [3, 4] or, when damping is included, in three-dimensional space [5]. Domains of vibrational stabilization of unstable systems lie outside the parametric resonance regions [6–8]. The boundaries of the parametric excitation zones are generally found by seeking approximate periodic solutions that separate the regions of stable and unstable motion. The most common asymptotic methods include the method of multiple scales [9], the averaging (van der Pol) method [10], the harmonic balance method, and others [11–13]. The method of stationarization of a periodic parameter [14], which we use in this work, is an extension of harmonic linearization. To apply it, the filter hypothesis for the linear part of the system must be satisfied, but it is not necessary to isolate a small parameter or assume slow variation of the oscillation envelope.

Parametric excitation can serve not only to stabilize oscillations of unstable objects but also to increase damping, thereby accelerating the stabilization of stable objects. Parametric anti-resonance is observed in systems with multiple degrees of freedom [15]. Its mechanism relies on coupling slow modal motions through parametric excitation. In accordance with Floquet theory, a monodromy matrix is constructed for the fundamental matrix of slow motion. Then the characteristic numbers (multipliers) are calculated for a fixed frequency and amplitude of the parameter. It was established that parametric excitation at a certain frequency (a combination of the object's natural frequencies) shifts the characteristic multipliers of the monodromy matrix toward the center of the unit circle [16]. The damping effect is significant for oscillations at the lowest elastic frequency. The damping system is implemented as open-loop control because the frequency of parametric excitation is calculated from the object model.

Another approach to increasing damping is implemented in a closed-loop system, where the damping properties depend on the balance of amplitudes and phases of the signals circulating in the loop. For linear closed-loop systems, these properties are known as stability margins and are determined by the Nyquist criterion at the crossover frequency. Increased damping in a closed-loop system can be achieved by creating excitation conditions that are similar to those of the first parametric resonance in terms of the frequency and amplitude of the parameter oscillations, but differ in the value of the time (phase) shift between the oscillations of the parameter and those of the output variable. When parametric resonance is excited, a specific shift is established. The stability margins of the closed-loop system can be adjusted by controlling this shift. Thus, parametric damping can be achieved instead of parametric resonance. The time shift between the periodic variation of the parameter and the oscillations of the output coordinate was used in [17, 18] to damp the output oscillations. It was found in [19] for vibrational stabilization of a flexible long rod under various boundary conditions that the basin of attraction of the vertical position depends significantly on the initial phase of the base oscillations.

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In this work, we investigate the efficiency of parametric damping in pendulum stabilization problems, implemented via a feedback control system. We propose a procedure for adjusting the variable coefficients of a parametric controller. The results of numerical simulation confirm the robustness of the stabilization algorithm.

Model of parametric controller in closed-loop stabilization system

Consider the problem of controlled stabilization of the stable equilibrium position of a pendulum. The objective of control is to improve the speed of the stabilization response. We take a linear model of a single-link pendulum as the plant. A fundamental feature of the plant is its light damping, which prevents the closed-loop system with a proportional controller from achieving acceptable performance.

The Mathieu equation describing the oscillations of a linear pendulum caused by a periodic variation of a parameter has the form

$$\ddot{\alpha}(t) + 2\nu\dot{\alpha}(t) + [1 + \mu\sin(kt - \varphi)]\alpha(t) = 0, \tag{1}$$

where t is the dimensionless time, $t = \omega_0$ (ω_0 is the natural frequency of the pendulum, is the dimensional time); α is the deviation of the pendulum from the equilibrium position, $\alpha^d = 0$; k is the ratio of the parameter oscillation frequency to ω_0; μ is the dimensionless amplitude of the parameter oscillations ($\mu < 1$); φ is the phase shift between the parameter oscillations and the pendulum motion; ν is the relative damping.

We rewrite Eq. (1) in the form

$$\ddot{\alpha} + 2\nu\dot{\alpha} + \alpha = -[\mu\sin(kt - \varphi)]\alpha = u \tag{2}$$

and regard the right-hand side of Eq. (2) as the control input u. Then the expression in square brackets is the parametric control law, and Eq. (2) describes the model of a closed-loop system.

Consequently, an unstable oscillatory process with the dimensionless frequency $k/2$, called the first parametric resonance, may occur in the closed-loop system. The goal of the study is to determine values of the variable controller coefficients that can suppress the output oscillations $\alpha(t)$. The quality of oscillation suppression in the closed-loop system is assessed by the settling time.

Let us determine the boundaries of the first parametric resonance in the closed-loop system. Suppose that in Eq. (2), there exist sustained oscillations of the variable

$$\alpha(t) = A\sin(k/2)t,$$

where A is the amplitude, $k/2$ is the dimensionless frequency.

Then the control law takes the form

$$u = -\mu A\cdot\sin[(k/2)t]\cdot\sin(kt - \varphi) = -0.5\mu A\{\cos[(k/2)t - \varphi] - \cos[(3k/2)t - \varphi]\}. \tag{3}$$

If $k/2 \approx 1$, i.e., close to the natural frequency of the oscillator, the second term on the right-hand side of Eq. (3) can be neglected, so after trigonometric transformations we obtain the approximate control law

$$u = -0.5\mu\{\sin\varphi + [(2/k)\cos\varphi]d\}\cdot A\sin(k/2)t \tag{3a}$$

where d is the differentiation operator with respect to dimensionless time.

Control law (3a) transforms the harmonic negative-feedback signal into a harmonic control signal of the same frequency $k/2$.

Since the equality

$$\{\sin\varphi + [(2/k)\cos\varphi]d\}|_{d=ik/2} = \exp\{i\,[(\pi/2) - \varphi)]\},$$

holds true, the model of the parametric controller for determining the resonance boundary can be written as a complex transfer coefficient [14]:

$$W(i\varphi) = (\mu/2)\cdot\exp\{i\,[(\pi/2) - \varphi]\}. \tag{4}$$

Consider a system closed by unity-gain negative feedback; its forward path consists of the parametric controller series-connected to the oscillator wtih a transfer function

$$H(p) = \frac{1}{p^2 + 2\nu p + 1}.$$

According to the Nyquist criterion, a periodic regime with the frequency $k/2$ is established in the closed-loop system assuming the equal magnitudes

$$|W(i\varphi)| = \left| H^{-1}(i\frac{\omega}{\omega_0}) \right|_{\frac{\omega}{\omega_0}=\frac{k}{2}},$$

therefore,

$$\frac{\mu}{2} = \sqrt{\left(1 - \frac{k^2}{4}\right)^2 + \nu^2 k^2}, \tag{5}$$

and the phases are also equal:

$$\pi/2 - \varphi - \text{arctg}[\nu k/(1 - k^2/4)] = -\pi. \tag{6}$$

For the case $k = 2$, Eqs. (5) and (6) give the values of the amplitude $\mu = 4\nu$ and phase shift $\varphi = \pi$ at which the closed-loop system is on the stability boundary. The procedure for adjusting the controller coefficients is as follows.

Step 1. Fix the parameter oscillation frequency at $k = 2$.

Step 2. Choose the parameter oscillation amplitude from the range $4\nu < \mu < 1$ within the parametric resonance region.

Step 3. Select the phase shift φ so as to provide a sufficient phase margin in the closed-loop system.

The question of whether the stabilization process can be accelerated compared to the uncontrolled process was answered by numerical experiment. The main goal in the numerical implementation of the parametric controller was to generate a signal with a prescribed oscillation period from accurate measurements of the damped output oscillations $\alpha(t)$. To this end, the output $\alpha(t)$ was processed in the Simulink environment by passing it successively through a quadratic function, a differentiator, and a pure delay block with the transfer function

$$H_{dl}(p) = \exp(-\tau p)$$

and then fed to a bistable relay [17]. The relay output produced a rectangular signal with a period half that of the oscillation 'period' of the output $\alpha(t)$ and phase-shifted by $\varphi = (\pi/2) - k\tau$.

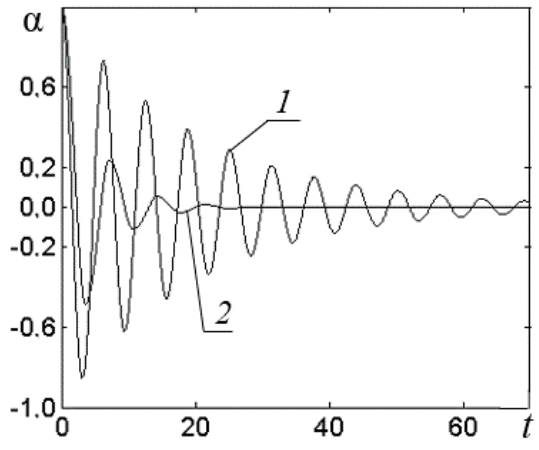


Fig. 1. Waveforms of free oscillations of the pendulum at relative damping $\nu = 0.05$ (curve *1*) and of the closed-loop system output $\alpha(t)$ at parametric controller coefficients $\mu = 0.5$ and $\tau = 0$ (curve *2*)

Fig. 1 shows the output waveform of the closed-loop system stabilizing the pendulum at the equilibrium position $\alpha^d = 0$ and the free oscillation waveform of the pendulum at $\alpha(0) = 1$. The settling time in the closed-loop system was reduced by a factor of four.

The stabilization efficiency is preserved when the amplitude μ and delay τ are varied over a wide range. It is recommended to choose μ closer to the upper bound for $\tau=0$. Thus, a distinctive feature of the parametric controller is its robustness.

Let us examine the performance of the parametric controller in the problem of suppressing an external harmonic disturbance. We add a harmonic disturbance torque to the right-hand side of Eq. (2). Replacing the periodic parameter by the complex coefficient (4) yields the approximate closed-loop equation:

$$\ddot{\alpha} + 2\nu\dot{\alpha} + \alpha + \frac{\mu}{2}\exp[i(\frac{\pi}{2} - \varphi)]\alpha = B\exp[i(\lambda t - \theta)], \tag{7}$$

where λ is the ratio of the disturbing torque frequency to the oscillator's natural frequency ω_0, B is the amplitude, θ is the disturbance phase.

The quality of suppression of forced oscillations is estimated from the frequency response of the closed-loop system. We seek the steady-state solution of Eq. (7) in the form

$$\alpha_\infty = A\exp\,[i\,(\lambda t - \psi)], \tag{8}$$

where A is the amplitude of the forced oscillations of the closed-loop output and ψ is the phase shift.

Note that the description of the parametric controller by the complex coefficient $W(i\varphi)$ becomes inadequate for $\lambda<1$, since the second term on the right-hand side of Eq. (3) cannot be neglected in the low-frequency range. Substituting solution (8) into Eq. (7), we obtain:

$$(B/A)\,\exp\,[i\,(\psi - \theta)] = H^{-1}(i\lambda) + W(i\varphi), \tag{9}$$

where $H^{-1}(i\lambda) = -\lambda^2 + 2\nu i\lambda + 1$.

The phase balance obtained from the graphical solution of Eq. (9) shows that a forced oscillation regime is fundamentally possible. The magnitude balance in Eq. (9) yields the frequency response of the closed-loop system:

$$\frac{A}{B} = \frac{1}{\sqrt{(1-\lambda^2 + \frac{\mu}{2}\sin\varphi)^2 + (2\nu\lambda + \frac{\mu}{2}\cos\varphi)^2}}. \tag{10}$$

If we set $\varphi=0$ in Eq. (10), which together with $4\nu<\mu<1$ provides the shortest settling time in the closed-loop system, the frequency response is reduced to

$$\frac{A}{B} = \frac{1}{\sqrt{(1-\lambda^2)^2 + (2\nu\lambda + \frac{\mu}{2})^2}}. \tag{11}$$

The closed-loop response at the natural frequency $\lambda = 1$ is

$$\left.\frac{A}{B}\right|_{\lambda=1} = \frac{1}{2\nu + \frac{\mu}{2}}$$

and decreases with increasing μ.

The dots in Fig. 2 mark the amplitudes of forced oscillations obtained from numerical experiments at $B = 1$. The frequency response (solid line) calculated from Eq. (11) is also shown for comparison.

The largest discrepancy is observed in the low-frequency range and is apparently due to the inadequate description of the parametric controller by the complex coefficient. Importantly, the closed-loop response at $\lambda=1$ is far below the resonance value of the linear oscillator, $A_{osc} = 1/2\nu$.

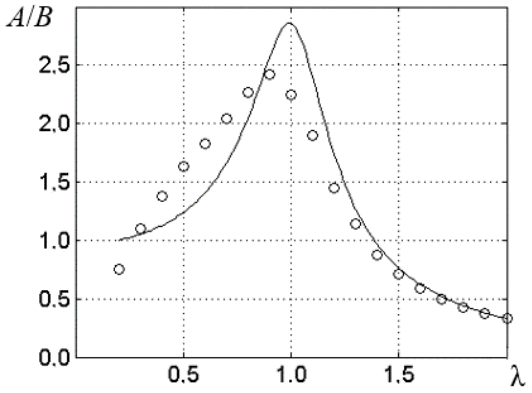


Fig. 2. Frequency responses of the closed-loop system obtained from Eq. (11) (solid line) and numerically (symbols). Amplitudes A of forced oscillations are calculated at $B = 1$, $\nu = 0.05$, $\mu = 0.5$, $\tau = 0$

Parametric controller as an addition to a conventional proportional controller

Consider the problem of controlled stabilization of an arbitrary position $\alpha^d \neq 0$ that does not coincide with the stable equilibrium position of the pendulum. Let us supplement the pendulum model with an actuator model, assumed to be a first-order lag element. The model of a closed-loop system with a proportional controller (P-controller) takes the form

$$(Td + 1)(d^2 + 2\nu d + 1)\alpha(t) = K_p[\alpha^d - \alpha(t)],$$

where d is the differentiation operator with respect to dimensionless time; $T = \omega_0$ is the dimensionless time constant of the lag element; K_p is the gain of the P-controller.

The efficiency of the P-controller in this problem is limited by the need to compromise between the requirements for the transient and steady-state (α_∞) regimes. To reduce the steady-state error

$$e_\infty = \alpha^d - \alpha_\infty = \alpha^d/(1 + K_p),$$

The gain K_p should be increased. To ensure the stability of a closed-loop system, the value of the coefficient K_p must satisfy the inequality

$$1 + K_p < (2\nu/T)\,(1 + 2\nu/T). \qquad (12)$$

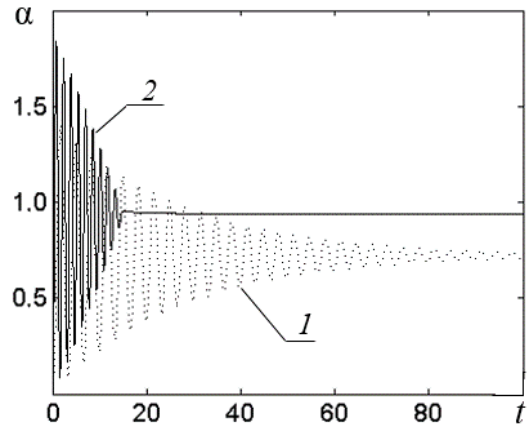


Fig. 3. Transient responses in the closed-loop system with the P-controller (gain K_p =2.6, curve *1*) and with the P-controller (gain K_p = 15) plus the parametric controller at μ=0.5 and τ=0 (curve *2*)

Fig. 3 shows the transient response ($\alpha^d = 1$) in the closed-loop system with the P-controller whose gain $K_p = 2.6$ provides a standard 60° phase margin in the closed-loop system for the chosen plant parameters $\nu = 0.05$, $T = 0.01$.

Let us include a parametric controller in the block diagram of the closed-loop system so that the controller's output is added to the P-controller output. The addition of the parametric controller makes it possible to achieve faster transient settling in the closed-loop system for the same K_p. Furthermore, it significantly extends (compared to condition (12)) the range of K_p values for which the closed-loop system is stable. This, in turn, allows reducing the steady-state error in the closed-loop system. Fig. 3 shows the transient response in the closed-loop system that includes the P-controller with K_p = 15 and the parametric controller. As a result, the steady-state error was reduced by a factor of 4.5 and the settling time by a factor of 5.6.

Stabilization of a planar two-link pendulum

Consider the problem of controlled stabilization of the stable equilibrium position of a planar two-link pendulum. We use a linear model of the two-link pendulum, assuming for simplicity that the end masses and the lengths of both links are equal. We write the model in dimensionless form (as was done in [20, 21]), introducing the natural frequency of a single-link pendulum ω_0 and a dimensionless time $t = \omega_0$:

$$M\begin{bmatrix}\ddot{\alpha}_1\\ \ddot{\alpha}_2\end{bmatrix}+2B\begin{bmatrix}\dot{\alpha}_1\\ \dot{\alpha}_2\end{bmatrix}+C\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix}=\begin{bmatrix}u\\ 0\end{bmatrix}, \tag{13}$$

where α_1 is the deviation of the first link from the vertical, $\alpha^d = 0$; α_2 is the deviation of the second link from the extension line of the first link; ν is the relative damping, assumed equal in both joints; M, B, C are the matrices of inertial, dissipative and quasi-elastic coefficients, respectively; u is the control input at the suspension point.

The square matrices take the form

$$M=\begin{bmatrix}5 & 2\\ 2 & 1\end{bmatrix},\ B=\begin{bmatrix}\nu & 0\\ 0 & \nu\end{bmatrix},\ C=\begin{bmatrix}3 & 1\\ 1 & 1\end{bmatrix}.$$

Since

$$B = \nu(2C - M),$$

the characteristic polynomial of model (13) has the form

$$x(p) = \det[M(p^2 - 2\nu p) + C(1 + 4\nu p)],$$

and, therefore,

$$x(p) = [p^2 - 2\nu p + (\omega_1)^2(1 + 4\nu p)]\cdot[p^2 - 2\nu p + (\omega_2)^2(1 + 4\nu p)], \tag{14}$$

where ω_1, ω_2 are the dimensionless natural frequencies of the double pendulum;

$$\omega_1 = \sqrt{2-\sqrt{2}} \approx 0.77,\ \omega_2 = \sqrt{2+\sqrt{2}} \approx 1.85.$$

Taking the variable $\alpha_1(t)$ as the output, we write the model of the plant

$$\alpha_1(p) = H_2(p)u(p),$$

where $H_2(p) = \dfrac{p^2 + 2\nu p + 1}{x(p)}$ is the transfer function.

The processes in the closed-loop system with the parametric controller

$$u = -[\mu \sin(kt - \varphi)]\alpha_1 \tag{15}$$

satisfy the equation

$$H_2^{-1}(d)\cdot\alpha_1 + [\mu\cdot\sin(kt - \varphi)]\cdot\alpha_1 = 0. \tag{16}$$

The range of admissible values of the dimensionless amplitude μ is calculated in the same way as for the Mathieu equation:

$$\mu < H_2^{-1}(0) = \omega_1^2\omega_2^2 = 2.$$

To determine the boundary of the first parametric resonance, we use the complex transfer coefficient (4) as an approximate model of the parametric controller. Varying μ, we calculate the frequencies that are roots of the equation

$$\left|W^{-1}(i\varphi)\right| = \frac{2}{\mu} = \left|H_2(i\frac{\omega}{\omega_0})\right|\Big|_{\frac{\omega}{\omega_0}=\frac{k}{2}}. \tag{17}$$

Eq. (17) may have four roots; therefore, the excitation region of the first parametric resonance for the two-link pendulum consists of two sub-regions near the doubled natural frequencies. Parametric resonance in the closed-loop system (16) occurs if the frequency of the parameter k is close to twice the first natural frequency ω_1 or twice the second natural frequency ω_2. It is convenient to determine the resonance boundaries using Bode diagrams:

$$L_2 = 20 \log|H_2(i\omega/\omega_0)|.$$

Fig. 4,*a* shows the Bode diagram L_2 at $\nu = 0.05$, which can have only two intersection points with the line

$$L_1 = 20 \log(2/\mu) \text{ at } \mu < 2.$$

Therefore, the excitation region of the first parametric resonance in the closed-loop system under consideration is located near the doubled first natural frequency. Fig. 4,*b* shows the boundary of the first parametric resonance.

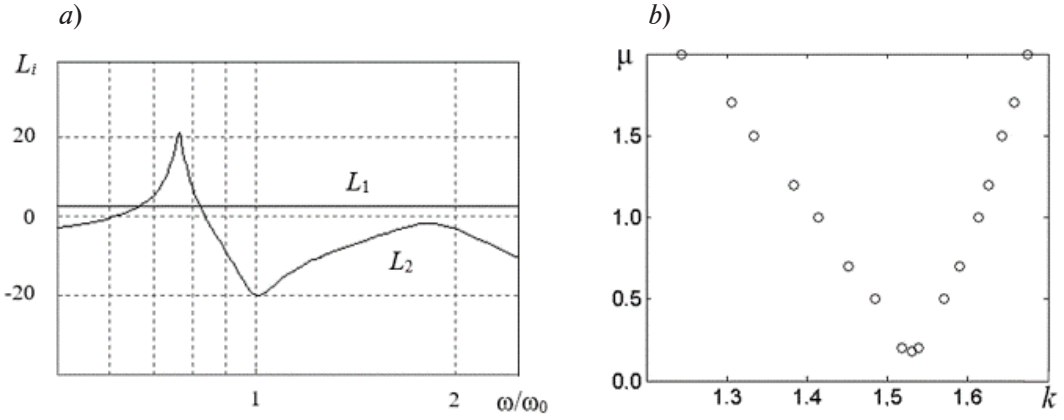


Fig. 4. Graphical representation of the solution of Eq. (17) for coefficients $\nu = 0.05$, $\mu = 1.5$ (*a*); dependence $\mu(k)$, which is the boundary of the first parametric resonance (*b*)

The parametric controller (15) is implemented approximately (as described above): for this reason, the measured angle $\alpha_1(t)$ is squared as a first step to generate a periodic signal. However, the free oscillations of the two-link pendulum for arbitrary initial conditions are a two-frequency process, since the roots of the characteristic polynomial (14) take the form

$$p_{12} = -(3-2\sqrt{2})\nu \pm i\sqrt{2-\sqrt{2}-(17-12\sqrt{2})\nu^2};$$

$$p_{34} = -(3+2\sqrt{2})\nu \pm i\sqrt{2+\sqrt{2}-(17+12\sqrt{2})\nu^2}.$$

Therefore, the variable $\alpha_1(t)$ is a superposition of exponentially decaying oscillations with frequencies depending on ν. For $\nu = 0.05$, the logarithmic decrement of damped oscillations with the second frequency is 14.2 times larger than that with the first frequency. Therefore, the main objective of parametric control is to damp the lower-frequency oscillations. It is expected that only the lower-frequency component remains in the measurements of $\alpha_1(t)$ in the closed-loop system (16) after a certain time. The parametric controller can efficiently damp the oscillations $\alpha_1(t)$ provided that the amplitude μ is sufficient and the phase shift φ ensures the stability of the closed-loop system.

Since no active control input is provided at the joint between the two links, the variable $\alpha_2(t)$ varies in accordance with the equation

$$\ddot{\alpha}_2 + 2\nu\dot{\alpha}_2 + \alpha_2 = -2\ddot{\alpha}_1 - \alpha_1.$$

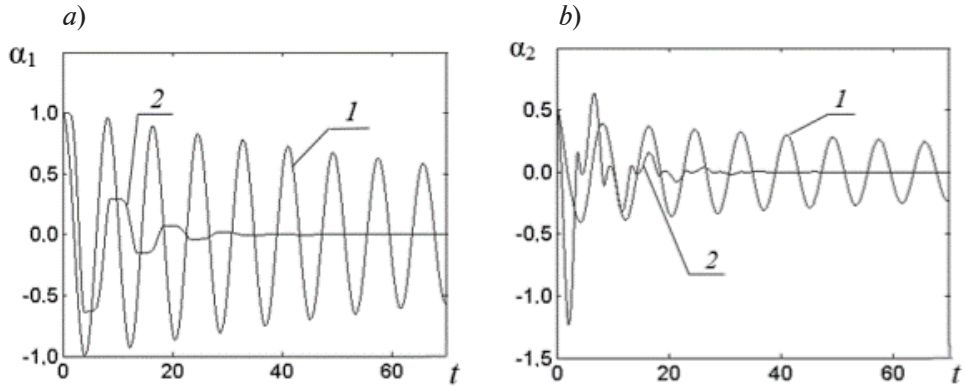


Fig. 5. Waveforms of variables α_1 (*a*) and α_2 (*b*):
free oscillations at $\nu = 0.05$ (curves *1*) and responses in the closed-loop system
with the parametric controller at $\mu = 1.5$, $\tau = 0$ (curves *2*)

The results of numerical simulation of the closed-loop system, which differs from Eq. (16) by the approximate implementation of the parametric controller, are shown in Fig. 5. To demonstrate the efficiency of the parametric controller, the free oscillation waveforms of the lightly damped two-link pendulum are also shown for comparison.

The settling time in the closed-loop system was significantly reduced compared to that of the free oscillations.

Conclusion

We examined the performance of a parametric controller incorporated into a closed-loop system designed to accelerate stabilization. A procedure for adjusting the controller's variable coefficients was proposed. Numerical simulation shows that exact tuning is not required to achieve parametric damping; it is sufficient to choose the amplitude and delay from the calculated range, while the frequency automatically doubles. We solved various stabilization problems for a lightly damped oscillator, where parametric control proved efficient.

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