

Conference materials

UDC 538.913

DOI: <https://doi.org/10.18721/JPM.184.105>

## Ising model on Fibonacci lattices: ring topology of sphere, cut ring, and torus

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**Abstract.** We study the Ising model on two-dimensional surfaces discretized using the Fibonacci method with Delaunay triangulation, considering the ring, cut ring, and torus topologies. The phase diagrams reveal a universal critical temperature of  $T_c \approx 3.33(3)J$  in the thermodynamic limit, which is consistent with the results for the Fibonacci sphere [1]. Despite the exclusion of topological defects (vertices with coordination numbers 5/7) in the ring and cut ring Fibonacci configurations, deviations from the critical temperature of the ideal flat triangular lattice are observed. The  $T_c$  values, similar to the spherical case, experience shifts. Notably, the torus, which possesses the minimal defect density (<1%), exhibits smooth convergence and negligible finite-size shifts in  $T_c$ . These results underscore the interplay between local connectivity and global topology in shaping critical phenomena.

**Keywords:** Ising model, topological defects, Fibonacci lattices, Monte Carlo simulation, phase diagrams

**Funding:** This study was funded by the grant from the Russian Science Foundation (RSF) No. FZNS-2024-0002.

**Citation:** Pochinok A.S., Molochkov A.V., Chernodub M.N., Chepak A.K., Ising model on Fibonacci lattices: ring topology of the sphere, cut ring and torus, St. Petersburg State Polytechnical University Journal. Physics and Mathematics. 18 (4.1) (2025) 31–36. DOI: <https://doi.org/10.18721/JPM.184.105>

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Материалы конференции

УДК 538.913

DOI: <https://doi.org/10.18721/JPM.184.105>

## Модель Изинга на решетках Фибоначчи: кольцевая топология сферы, кольцо с разрезом и тор

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**Аннотация.** Мы исследуем модель Изинга на двумерных поверхностях, дискретизированных методом Фибоначчи с триангуляцией Делоне, рассматривая топологии кольца, кольца с разрезом и тора. Фазовые диаграммы демонстрируют универсальную критическую температуру  $T_c \approx 3,33(3)J$  в термодинамическом пределе, что согласуется с результатами для сферы Фибоначчи [1]. Несмотря на исключение топологических дефектов (вершин с координационными числами 5/7) в конфигурациях кольца и кольца с разрезом Фибоначчи наблюдаются отклонения от критической температуры идеальной плоской треугольной решетки. Значения  $T_c$  аналогично случаю

сферы, испытывают сдвиги. Примечательно, что тор, обладающий минимальной плотностью дефектов (<1%), демонстрирует плавную сходимость и пренебрежимо малые сдвиги  $T_c$ . Эти результаты подчеркивают взаимосвязь между локальной связностью и глобальной топологией в формировании критических явлений.

**Ключевые слова:** модель Изинга, топологические дефекты, решетки Фибоначчи, Монте-Карло моделирование, фазовые диаграммы

**Финансирование:** Работа выполнена при поддержке гранта № FZNS-2024-0002 Министерства науки и высшего образования Российской Федерации.

**Ссылка при цитировании:** Починок А.С., Молочков А.В., Чернодуб М.Н., Чепак А.К. Модель Изинга на решетках Фибоначчи: кольцевая топология сферы, кольцо с разрезом и тор // Научно-технические ведомости СПбГПУ. Физико-математические науки. 2025. Т. 18. № 4.1. С. 31–36. DOI: <https://doi.org/10.18721/JPM.184.105>

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## Introduction

The two-dimensional Ising model is a fundamental system for studying phase transitions and critical phenomena [2]. Although its characteristics on standard flat lattices are thoroughly documented, the behavior on curved surfaces featuring non-trivial topology introduces novel theoretical questions and research avenues.

This research examines the ferromagnetic Ising model on curved surfaces, where the lattices are defined by a Fibonacci distribution and discretized using Delaunay triangulation [3]. This technique produces highly uniform tessellations consisting of nearly equal-area triangles [4], offering a robust approach for investigating the thermodynamic limit on curved geometries. A defining aspect of such lattices is the emergence of topological defects—specifically vertices with coordination numbers five and seven. Growth in the number of nodes within a Fibonacci sphere induces discrete transitions in the triangulation structure, marking sudden shifts in network connectivity. These structural changes significantly affect the system's statistical mechanics, giving rise to discontinuities reminiscent of first-order transitions as the sphere expands [1].

Fibonacci lattices have already been used in studies of the XY model [5] and have also proven effective in modeling magnetic resonance [6], among other applications. In turn, studying the ferromagnetic phase transition on curved surfaces will become an important aspect for modeling spherical nanoparticles and magnetic films.

## Fibonacci lattices

Fibonacci lattices are characterized by an almost uniform and isotropic distribution of nodes. This allows for efficient discretization of curvilinear surfaces using triangles of approximately equal area. The structure is based on Fibonacci numbers. A Fibonacci number  $F$  is part of an infinite sequence in which each element, starting from the third ( $i \geq 3$ ), is the sum of the two previous elements:  $F_i = F_{i-1} + F_{i-2}$ . The first few numbers in the sequence are: 0, 1, 1, 2, 3, 5, 8, 13, 21... In the limit of a large number of elements, the ratio of two consecutive numbers approaches the value of the golden ratio  $g = (1 + \sqrt{5})/2$ .

The Fibonacci distribution on two-dimensional surfaces forms a structure consisting of two sets of spiral arcs [4, 7]. In these arcs, the distance between neighboring points is determined by Fibonacci numbers [8]. Some sets of spirals are twisted clockwise, while others are twisted counterclockwise. As the lattice radius (total number of nodes) increases, the number of dominant spirals grows. It can also be shown that the Fibonacci lattice is essentially the result of a sequential arrangement of elements along a single “generative spiral” [9]. The angular distance between elements of this generative spiral corresponds to  $g$ .

When discretizing Fibonacci surfaces using a set of triangles in our case, Delaunay triangulation – defects in connectivity properties are observed, as detailed for the sphere case in [1]. The number of neighboring vertices varies from 5 to 7, with points having 6 neighbors being predominant.

To examine the influence of lattice connectivity defects on the critical temperature of the Ising model, it is necessary to study the phase transition in a region where the number of 5- and 7-coordinated nodes tends to 0. Defective connectivity is primarily localized at the poles of the spherical lattice. Therefore, we use a ring topology of the Fibonacci sphere. This is achieved by excising the poles, i.e., the polar regions with a high concentration of coordination number defects. Such a structure forms an equatorial zone dominated by hexagonal order (6 neighbors), bringing it closer to a regular triangular lattice. The width of the equatorial region with 6 neighbors depends on the number of lattice sites  $N$ . The ring topology obtained by excising the poles of the triangulated sphere is shown in Fig. 1, *a*.

To most closely approximate the case of an ideal flat triangular lattice for result comparison, it is necessary to obtain a flat space with a Fibonacci distribution. In this work, we achieve such a configuration by removing nodes along one generatrix of the ring. Thus, we obtain a locally flat space (for all points except boundary ones) with a Fibonacci distribution with a hexagonal structure (Fig. 2, *b*).

A toroidal triangulated lattice is equivalent to an ideal flat triangular lattice with periodic boundary conditions, for which the known critical temperature of the Ising model is  $T_c \approx 3.64$ . Therefore, we investigate the phase transition on the Fibonacci torus (Fig 1, *c*). Similar to the sphere, the Fibonacci torus has points with 5 and 7 neighbors, but their number is significantly smaller (<1%) (Fig. 2). The Fibonacci torus was defined by equations (1, 2):

$$x = (R + r \cos \varphi) \cos \theta, \quad y = (R + r \cos \varphi) \sin \theta, \quad z = r \sin \varphi, \quad (1)$$

$$\varphi = \frac{2\pi k}{g}, \quad \theta = \frac{2\pi k}{N}, \quad (2)$$

where  $k$  is the site index and  $N$  is the number of sites in the lattice.

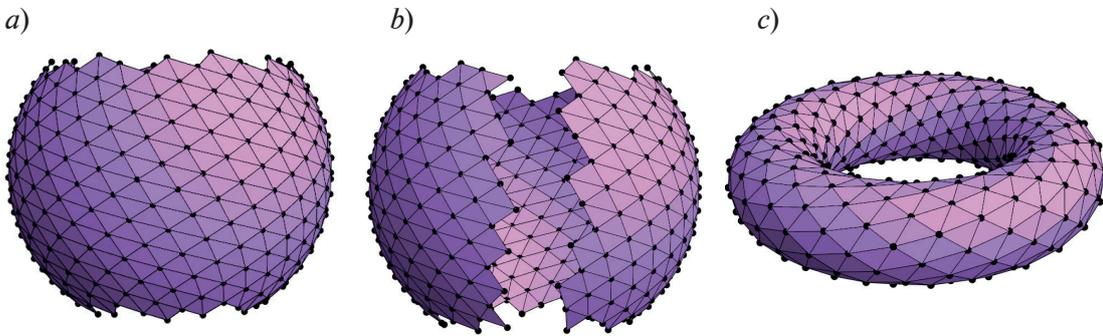


Fig. 1. Fibonacci lattices: ring (*a*); cut ring (*b*); torus (*c*)

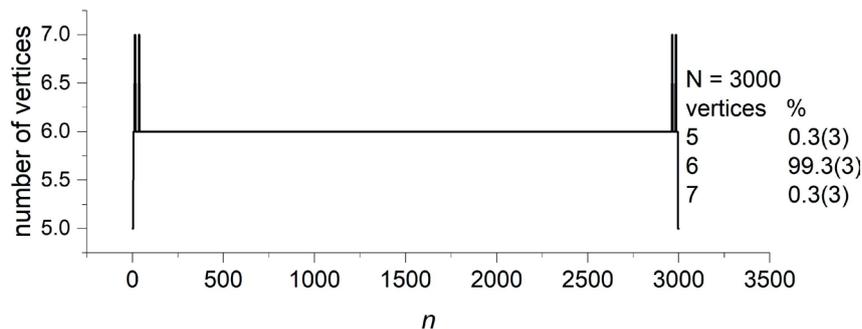


Fig. 2. Distribution of the number of neighboring vertices for a torus with 3000 and 47000 nodes

The Fibonacci torus exhibits interesting patterns related to its spiral structure. The Euler characteristic of the Fibonacci torus equals zero only for specific filling ranges. For example, in the filling range  $2500 < N < 3000$ , the lattices are topologically equivalent to a torus. For  $N < 2500$ , the Euler characteristic is not zero, indicating defects such as intersections of triangle bonds. Further statistical analysis revealed that the “true” torus occurs sequentially and was found for  $N = \{7000 - 8000; 17000 - 23000; 46000 - \dots\}$ . Moreover, as the number of points increases, the width of the  $N$  range for “true” torus increases. This effect, similar to the sphere, may be associated with the formation of new spirals but requires more detailed study.

### Ising model

The two-dimensional Ising model is one of the simplest statistical models, where the spin variable  $S_x$  represents the magnetic moment of an atom at site  $x$ . This variable can take values  $S_x = +1$  or  $S_x = -1$ , corresponding to upward or downward orientation of the magnetic moment relative to the atomic crystalline plane. The Hamiltonian of the system is given by (3):

$$H = -J \sum_{\langle x,y \rangle} S_x S_y - h \sum_x S_x. \quad (3)$$

We consider dimensionless variables in units of the coupling  $J = 1$ . All calculations will be performed for systems in the absence of an external magnetic field  $h = 0$ .

A parameter critically sensitive to the phase transition is the magnetic susceptibility  $\chi$ , which quantitatively characterizes fluctuations of the order parameter – spontaneous magnetization.

Within the formalism of the canonical ensemble and in the absence of an external magnetic field, the magnetic susceptibility is defined via fluctuations of the total magnetization  $M$  by the following relation:

$$\chi(T, N) = \frac{\langle M^2(T, N) \rangle - \langle M(T, N) \rangle^2}{T}, \quad (4)$$

where  $M = \sum_x S_x$  is the total magnetic moment of the system.

### Results and Discussion

The study of the Ising model phase transition on a ring lattice was conducted with fixed boundary conditions: spins at the ring boundaries were fixed in the +1 state. This spin fixation minimizes boundary fluctuations, simplifying the analysis of the contribution from internal nodes to the phase transition.

Numerical simulation was carried out using the Monte Carlo method, using the Metropolis algorithm.

The magnetic susceptibility of the Fibonacci ring (Fig. 3, *b* (right)) was calculated using formula (4). This effect arises because the size of the excised poles and the number of boundary points are related to the neighbor distribution. Critical temperatures for the three sequences, derived from Gaussian function approximation of the magnetic susceptibility peaks, are shown in Fig. 4. The magnetic susceptibility exhibits behavior similar to the sphere case [1], presenting analogous sequences. In the continuum limit, the critical temperature of the phase transition on the Fibonacci ring approaches that of the sphere,  $T_c \approx 3.3$ .

For a comparative analysis of the critical temperature behavior of the ferromagnetic phase transition between Fibonacci lattices and a flat triangular lattice, the magnetic susceptibility was calculated for the cut ring topology (Fig. 1, *b*). Such a lattice is considered to be locally flat if we consider the regions excluding the boundaries. The boundary conditions were similarly fixed.

The behavior of the magnetic susceptibility (Fig. 3, *a* (left)) is analogous to the ring case and presents itself as sequences. The plot of the critical temperature dependence on the inverse number of nodes is shown in (Fig. 4). For all sequences, the critical temperature differs insignificantly and, as in previous Fibonacci lattices, approaches  $T_c \approx 3.3$  in the limit.

For further investigation of the Ising model phase transition on Fibonacci lattices, topologically correct torus with Euler characteristic equal to zero were selected, namely with  $N = \{3000, 8000, 17000, 47000\}$ . Using the same calculation methods, magnetic susceptibility was obtained for this set of lattices (Fig. 5).

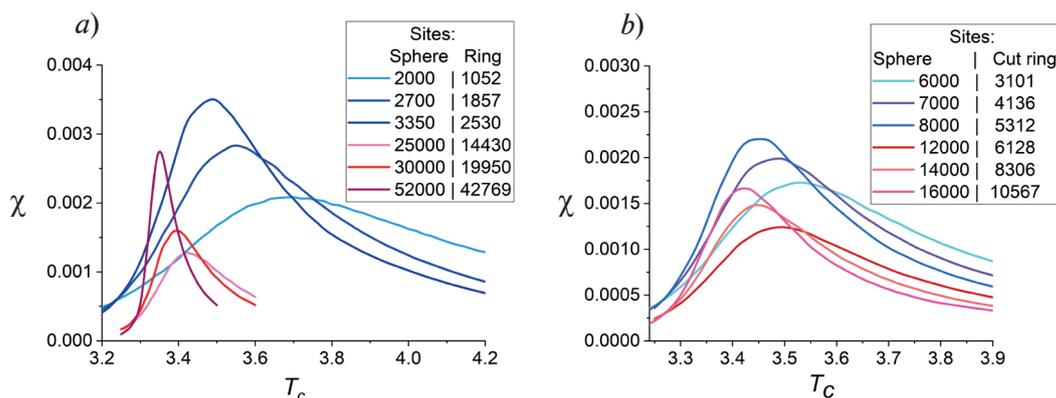


Fig. 3. Examples of two sequences of magnetic susceptibilities on Fibonacci ring (*a* – left) and Fibonacci cut ring (*b* – right) lattices. Data corresponding to two different sequences are highlighted with shades of similar colors

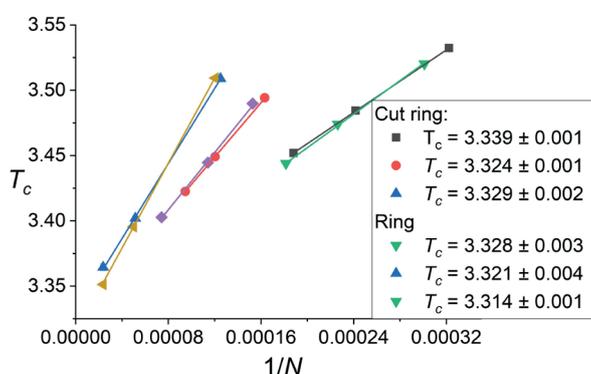


Fig. 4. Scaling of the pseudocritical temperature derived from the magnetic susceptibility approximation on the Fibonacci ring and cut ring as a function of the inverse number of sites,  $1/N$

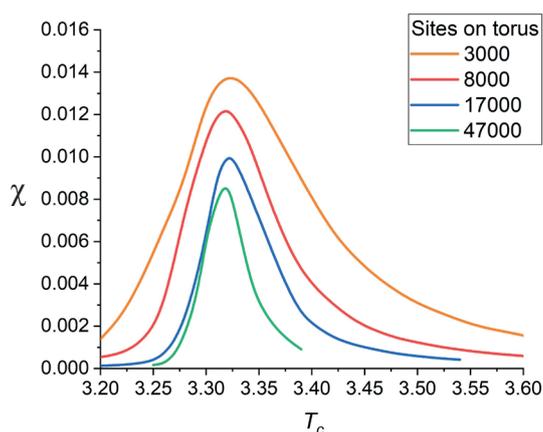


Fig. 5. Scaling of the pseudocritical temperature derived from the magnetic susceptibility approximation on the Fibonacci torus as a function of the inverse number of nodes,  $1/N$

Unlike the sphere and its ring topologies, on Fibonacci torus the phase transition temperature is consistently  $T_c \approx 3.2$ . The value is constant due to the absence of abrupt changes in connectivity properties.

Since the torus corresponds to a flat triangular lattice with periodic boundary conditions, it was expected that the critical temperature would be close to  $T_c \approx 3.64$  [10]. However, contrary to expectations, the critical point of the Ising model on the Fibonacci torus is close to the values on the sphere, ring, and Fibonacci cut ring.

### Conclusion

In summary, it can be concluded that the Ising model on Delaunay-triangulated Fibonacci lattices demonstrates universal critical temperature behavior regardless of topology. The proximity of the  $T_c \approx 3.33(3)J$  value to that of the flat triangular lattice emphasizes the role of quasi-isotropic node distribution and the predominance of hexagonal order. However, the slight deviation is due to the spiral structure of the Fibonacci distribution, which leads to a decrease in  $T_c$  compared to theoretical predictions for regular lattices.

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*Received 16.09.2025. Approved after reviewing 01.12.2025. Accepted 04.12.2025.*