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Qubit-qubit entanglement in the Tavis–Cummings model with two independent resonators

A.R. Bagrov ✉ [1], E.K. Bashkirov [1]

[1] Samara National Research University, Samara, Russia

✉ alexander.bagrov@mail.ru

Abstract. In this paper, a three-qubit Tavis–Cummings model with two independent lossless single-mode resonators is considered. It is assumed that the initial states of the resonator fields are thermal fields and the qubits are in genuine entangled W- and GHZ-type states. For the model under study and the specified initial states of the resonator fields and qubits, we have exactly solved the quantum Liouville equation for the full density matrix. The full density matrix was used to calculate the entanglement parameters – negativity and fidelity. The computer simulation results showed that in the investigated model, the entanglement for all initial qubit states breaks down rapidly with increasing intensity of the thermal fields of the resonators compared to the previously investigated three-qubit models. Moreover, a sudden death of entanglement is observed even for vacuum resonator fields.

Keywords: independent resonators, fidelity, qubits, negativity, sudden death of entanglement, single-photon transitions, thermal field, entanglement

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Материалы конференции
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Кубит-кубитное перепутывание в модели Тависа–Каммингса с двумя независимыми резонаторами

А.Р. Багров ✉ [1], Е.К. Башкиров [1]

[1] Самарский национальный исследовательский университет, г. Самара, Россия

✉ alexander.bagrov@mail.ru

Аннотация. В работе рассматривается трехкубитная модель Тависа–Каммингса с двумя независимыми одномодовыми резонаторами без потерь. Предполагается, что начальным состоянием полей резонаторов являются тепловые поля, а кубиты находятся в истинно перепутанных состояниях W- и GHZ-типа. Для исследуемой модели и указанных начальных состояний полей резонаторов и кубитов нами было точно решено квантовое уравнение Лиувилля для полной матрицы плотности. Полная матрица плотности использовалась для вычисления параметров перепутывания – отрицательности и степени совпадения. Результаты компьютерного моделирования показали, что в исследуемой модели перепутанность для всех начальных состояний кубитов быстро разрушается с увеличением интенсивности тепловых полей резонаторов по сравнению с ранее исследуемыми трехкубитными моделями. Более того, даже для вакуумных полей резонаторов наблюдается эффект мгновенной смерти перепутывания.

Ключевые слова: независимые резонаторы, степень совпадения, кубиты, отрицательность, мгновенная смерть перепутывания, однофотонные переходы, тепловое поле, перепутанность

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Introduction

Multiqubit genuine entangled qubit states are the foundation for various areas of quantum technology [1, 2]. In particular, entangled qubit states are used in quantum computers. The elementary cells of any quantum computer are so-called gates, which allow the performance of various operations on qubits, i.e., the building of various quantum algorithms. Descriptions of physical realizations of two- and three-qubit gates can be found, for example, in [3]. The increased interest in three-qubit systems is explained by the fact that they can be used for the creation of universal three-qubit gates of the Toffoli or Fredkin type, which can significantly simplify codes of quantum error corrections and will allow the creation of more complex quantum algorithms for solving various problems. Despite the large number of experimental and theoretical works devoted to the study of entangled states, many aspects of the entanglement, especially in the case of multiqubit systems, need further detailed study. One of the serious problems arising when using entangled states in quantum information processing problems is the inevitable degradation of entangled states and the sudden death of entanglement [4, 5]. These effects have been studied in detail in various two-qubit systems, and much less work is devoted to the study of sudden death of entanglement with respect to multiqubit systems (see refs. in [6]).

In this paper, we study in detail the dynamics of qubit entanglement for a three-qubit model in which three qubits are trapped in two independent resonators and interact with the corresponding mode of the resonator field via single-photon processes. For the model under study we have solved exactly the quantum Liouville equation for the full density matrix describing the “three qubits+two modes of the resonator field”. This density matrix was used to calculate the fidelity, the negativity criterion, and to analyze the dynamics of qubit entanglement.

The Tavis–Cummings model and its solution

In this work, the dynamics of entanglement of three identical qubits Q_1, Q_2 and Q_3 are investigated. Qubit Q_1 is trapped in the resonator *a*, qubits Q_2 and Q_3 are in the resonator *b*. Qubits resonantly interact with the quantum thermal field mode of their ideal resonator via single-photon processes. The interaction Hamiltonian of the model under study in standard notations and approximations is written in the following form

$$H_{Int} = \hbar\gamma\left(\sigma_{Q_1}^{+}\xi + \sigma_{Q_1}^{-}\xi^{+} + \sigma_{Q_2}^{+}\eta + \sigma_{Q_2}^{-}\eta^{+} + \sigma_{Q_3}^{+}\eta + \sigma_{Q_3}^{-}\eta^{+}\right), \quad (1)$$

where $\sigma_l^{+} = |+\rangle_{ll}\langle -|$ and $\sigma_l^{-} = |-\rangle_{ll}\langle +|$ are the transition operators between the excited $|+\rangle_l$ and the ground $|-\rangle_l$ states in the *l*-th qubit ($l = Q_1, Q_2, Q_3$), $\xi^{+}(\eta+)$ and $\xi(\eta)$ are the creation and the annihilation operators of the photons in the mode of the resonator $a(b)$, γ is the qubit-photon coupling.

As the initial state of the resonator field, we choose a thermal states with a density matrixes of the form:

$$\wp_{F_a}(0) = \sum_{n_a} p_{n_a} |n_a\rangle\langle n_a|, \ \wp_{F_b}(0) = \sum_{n_b} p_{n_b} |n_b\rangle\langle n_b|, \quad (2)$$

Here, there are weight coefficients $p_{n_i} = \frac{\langle n \rangle_i^{n_i}}{(1+\langle n \rangle_i)^{n_i+1}}$, $\langle n \rangle_i = \frac{1}{e^{\hbar\omega_i/k_B T_i} - 1}$ is the mean number of photons in the i-th resonator ($i = a$, b), T_i is the cavity temperature. Let the initial states of qubits be the W-states, such as

$$|W_1(0)\rangle_{Q_1Q_2Q_3} = \cos\theta\,|+,+,-\rangle + \sin\theta\sin\varphi\,|+,-,+\rangle + \sin\theta\cos\varphi\,|-,+,+\rangle, \tag{3}$$

$$|W_2(0)\rangle_{Q_1Q_2Q_3} = \cos\theta\,|-,-,+\rangle + \sin\theta\sin\varphi\,|-,+,-\rangle + \sin\theta\cos\varphi\,|+,-,-\rangle, \tag{4}$$

or GHZ-state

$$|G(0)\rangle = \cos\phi\,|+,+,+\rangle + \sin\phi\,|-,-,-\rangle. \tag{5}$$

Here θ, φ, and ϕ are the parameters that determine the initial degree of qubit entanglement.

We derived the solutions of the quantum Liouville equation for the whole density matrix $\wp_{Q_1Q_2Q_3F_aF_b}$ of the considered system with Hamiltonian (1) and the indicated initial states for the qubits and the resonator field (2)-(5)

$$i\hbar\frac{\partial \wp_{Q_1Q_2Q_3F_aF_b}}{\partial t} = \left[H_{Int}, \wp_{Q_1Q_2Q_3F_aF_b}\right]. \tag{6}$$

Even compact solutions for the quantum Liouville equation (6) look too large and for this reason are not presented in this paper to save room.

In this paper, our focus was on two parameters of entanglement: pairwise negativity $\varepsilon_{Q_iQ_j}(\gamma t)$ and fidelity $F(\gamma t)$. The fidelity is written as follows [7]: $F\left[\wp(0), \wp'(t)\right] = Tr\left[\wp(0)\wp'(t)\right]$, where $\wp(0) \equiv \wp_{Q_1Q_2Q_3}(0)$ is the initial three-qubit density matrix, $\wp'(t) \equiv \wp_{Q_1Q_2Q_3}(t)$ is the three-qubit density matrix at subsequent time instants t. The three-qubit density matrix can be obtained from the full density matrix by averaging over the field variables of the two resonators $\wp_{Q_1Q_2Q_3} = Tr_{F_a}Tr_{F_b}\wp_{Q_1Q_2Q_3F_aF_b}$.

We define the negativity $\varepsilon_{Q_iQ_j}(\gamma t)$ in a standard way [8]: $\varepsilon_{Q_iQ_j} = -2\sum \lambda_{l,ij}^{-}$, where $\lambda_{l,ij}^{-}$ are the negative eigenvalues of the two-qubit density matrixes partially transposed in variables of one qubit $\wp_{ij}^{T_1}$. To calculate the negativity of two qubits, we need to compute a partially transposed reduced two-qubit density matrix over the variables of one qubit $\wp_{Q_iQ_j}^{T_1}$, whose elements are defined as follows via the two-qubit density matrix $\left\langle p_i, m_j \middle| \wp_{Q_iQ_j} \middle| k_i, l_j \right\rangle^{T_1} = \left\langle k_i, m_j \middle| \wp_{Q_iQ_j} \middle| p_i, l_j \right\rangle$, where $|p\rangle, |m\rangle, |k\rangle, |l\rangle = |+\rangle, |-\rangle$ and two-qubit density matrix $\wp_{Q_iQ_j} = Tr_{Q_x}\wp_{Q_1Q_2Q_3}$ $(i, j, x = 1,2,3;\ i \neq j \neq x)$.

Results and Discussion

The results of computer modeling of the pairwise negativities $\varepsilon_{Q_iQ_j}(\gamma t)$ and fidelity $F(\gamma t)$ for initial qubit W-state (3) and thermal field (2) are shown in Fig. 1. Figures represent the behavior of negativities, calculated for various mean photon numbers $\langle n \rangle$ with the initial parameters $\theta = \arccos(1/\sqrt{3})$, $\varphi = \pi/4$. In Fig. 2, we plot the fidelity for the initial qubit GHZ-state (5) and thermal field (2) in the model (1) and model "three qubits in common resonator" with the initial parameters $\phi = \pi/4$.

The following conclusions can be drawn from the presented figures. From Figs. 1 and 2, we can clearly see that the maximum degree of entanglement decreases monotonically with increasing intensity of the thermal fields of the resonators $\langle n \rangle$ for any initial states of qubits (3)-(5). From the comparison of Fig. 1, *a* and 1, *b*, we can conclude that the Q_2 and Q_3 qubits are more robust to the thermal noise of the resonator than the Q_1 and Q_2 (or Q_1 and Q_3) qubits. Moreover, the maximum entanglement degree of qubits Q_2 and Q_3 can exceed the initial pairwise entanglement of qubits $\varepsilon_{Q_iQ_j} = (\sqrt{5}-1)/3$ for the W-states (3)-(4). Moreover, from Fig. 1, *a* and Fig. 1, *b* we can clearly see that at some moments of time negativity takes zero values ($\varepsilon_{Q_iQ_j}(\gamma t) = 0$). This suggests the presence of an sudden death effect of entanglement qubits for the W-state (3) even in the case of low intensities of the thermal fields of the resonators $\langle n_a \rangle = \langle n_b \rangle = 0.001$. The entangled W-type states (3)-(4) behave identically and for this reason the plots for the W-state (4) are not given. An analysis of the behavior of the fidelity (see Fig. 1, *c* and Fig. 2, *a*) shows that the initial genuine entangled W-states (3)-(4) or GHZ-state (5) never return to the initial states during the evolution process.

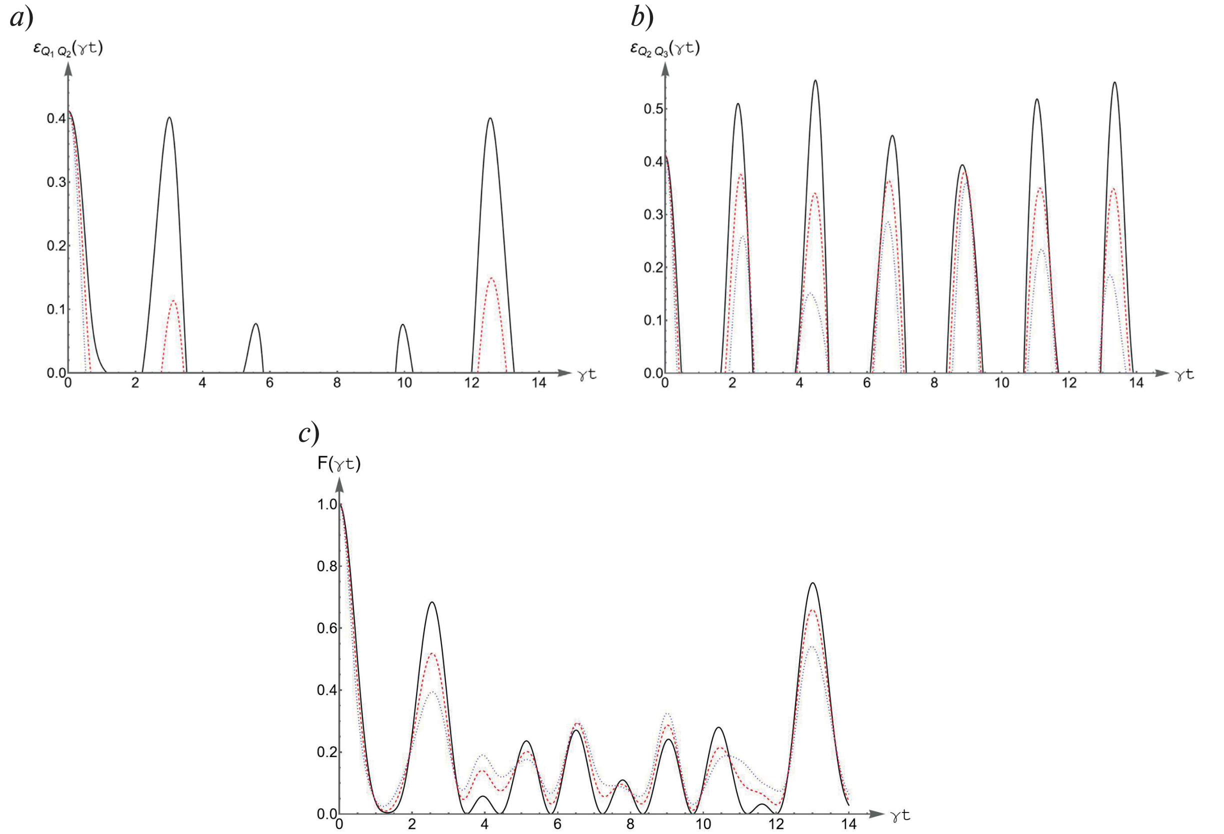

Fig. 1. The negativity $\varepsilon_{Q_iQ_j}(\gamma t)$ (*a*, *b*) and fidelity $F(\gamma t)$ (*c*) as functions of the scaled time γt for the initial W-state of the form (3). The mean number of photons: $\langle n_a \rangle = \langle n_b \rangle = 0.001$ (black solid line), $\langle n_a \rangle = \langle n_b \rangle = 0.2$ (red dashed line), $\langle n_a \rangle = \langle n_b \rangle = 0.5$ (blue dotted line). Initial parameters: $\theta = \arccos(1/\sqrt{3}),\ \varphi = \pi/4$

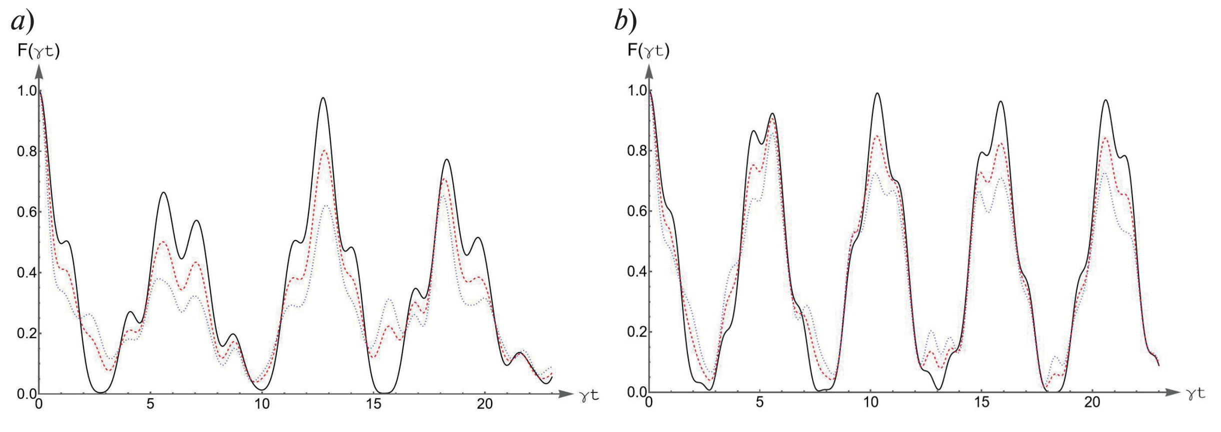

Fig. 2. The fidelity $F(\gamma t)$ as a function of the scaled time γt for the initial GHZ-state of the form (5) in the model (1) (*a*) and in the model "three qubits in common resonator" (*b*). The mean number of photons: $\langle n_a \rangle = \langle n_b \rangle = 0.001$ (black solid line), $\langle n_a \rangle = \langle n_b \rangle = 0.2$ (red dashed line), $\langle n_a \rangle = \langle n_b \rangle = 0.5$ (blue dotted line). Initial parameter: $\phi = \pi / 4$

Thus, the initial entanglement is completely destroyed even in the case of the vacuum field of the resonator $(\langle n \rangle \to 0)$. From the comparison of Fig. 2, *a* and Fig. 2, *b*, it can be seen that the entanglement of qubits in the model with three qubits in a common resonator is more robust to the thermal noise of the resonator than the model studied in this paper. A similar result is obtained for genuine entangled W-type states (3)-(4) (see refs. [9]).

Conclusion

Thus, in this paper we have exactly solved the quantum Liouville equation for the three-qubit Tavis–Cummings model in which one of the qubits Q_1 is in the resonator *a,* and the two remaining qubits Q_2 and Q_3 are trapped in the second resonator *b*. It is assumed that all qubits are identical and their coupling to the field is equal. Our focus was on the genuine entangled Werner and Greenberger–Horn–Zeilinger states, and the fields of the resonators are in the thermal state. The computational results show that in the model under study, the genuine entangled states are quite fragile with respect to the thermal noise of the resonators compared to the previously studied three-qubit models [9]. We also show, using the fidelity $F(\gamma t)$, that the W- and GHZ-type states never return to the initial states during evolution, even for vacuum fields of $(\langle n \rangle \to 0)$ resonators. Moreover, the evolution of the negativity criterion shows that the sudden death of pair entanglement occurs for both W-states (3)-(4).

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THE AUTHORS

BAGROV Alexander R.
alexander.bagrov@mail.ru
ORCID: 0000-0002-1098-0300

BASHKIROV Eugene K.
bash@ssau.ru
ORCID: 0000-0001-8682-4056

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