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FEATURES OF USING THE MAIN CRITERION OF THE FDD METHOD IN IDENTIFYING THE DYNAMIC CHARACTERISTICS OF STRUCTURES

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Abstract. This publication continues the series of articles devoted to the Frequency Domain Decomposition (FDD) method that serves to identify the dynamic characteristics of structures (natural frequencies and forms of natural vibrations) from the vibration tests data without using vibroexcitation. The method is based on the SV-decomposition of the cross-spectral density matrix (CSDM) of the measured signals. The proof of the possibility of using the FDD for an incomplete CSDM was given, such matrices being typical when working with large-sized structures. In this case, vibration tests include two types of measurements: continuous long-term ones taking at one or more selected reference points and consecutive short ones at other measuring points. The presented results may be useful in the operation of large concrete dams and other hydraulic structures to identify their dynamic characteristics.

Keywords: decomposition in frequency domain, dynamic characteristics, matrix of mutual spectral densities

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ОСОБЕННОСТИ ИСПОЛЬЗОВАНИЯ ОСНОВНОГО КРИТЕРИЯ МЕТОДА FDD ПРИ ИДЕНТИФИКАЦИИ ДИНАМИЧЕСКИХ ХАРАКТЕРИСТИК СООРУЖЕНИЙ

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Аннотация. Данная публикация продолжает цикл статей, посвященных применению метода декомпозиции в частотной области (FDD), который служит для идентификации динамических характеристик сооружений (собственных частот и форм собственных колебаний) по данным вибрационных испытаний, без применения вибровозбуждения. Метод основан на сингулярном разложении матрицы взаимных спектральных плотностей (МВСП) измеренных сигналов. В статье приведено доказательство возможности использования FDD для неполной МВСП. Такие МВСП характерны при работе с крупными сооружениями. В этом случае вибрационные испытания включают два вида измерений: непрерывные длительные в одной или нескольких выбранных опорных точках и последовательные короткие в остальных измерительных точках. Представленные результаты могут быть полезными при эксплуатации больших бетонных плотин и других гидротехнических сооружений для идентификации их динамических характеристик.

Ключевые слова: декомпозиция в частотной области, динамические характеристики сооружения, матрица взаимных спектральных плотностей

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Introduction

Determining the dynamic characteristics of structures (natural frequencies, eigenmodes, attenuation decrements) is an important problem in the construction industry and, in particular, for hydraulic structures (HS) in seismic areas, since their operational safety requirements are particularly high.

The methods that allow experimentally determining the dynamic characteristics of structures under normal operating conditions are collectively known as operational modal analysis (OMA). It should be noted that using OMA is possible in the case when the operational dynamic effects on the structure have a stationary broadband frequency behavior close to white noise.

This article discusses one of the most commonly used methods from the OMA group, internationally known as Frequency Domain Decomposition (FDD). This method is based on singular-value decomposition of the cross-spectral density matrix (CSDM) of simultaneously performed measurements. It is described in detail in [3–5, 8].

An important feature of the method is the presence of a formalized criterion for identifying natural frequencies (the term ‘criterion’ is used for brevity from now on), specifically that as a function of frequency, the first singular value of the CSDM should have local maxima near the natural (modal) frequencies [1, 3, 5, 7].

Since 2019, the FDD method have been adopted since 2019 by scientists at B.E. Vedeneev All-Russian Research Institute of Hydraulic Engineering (St. Petersburg, Russia) for instrumental identification of the dynamic characteristics of the hydraulic structures (HS) of various hydroelectric power plants (HPP), in particular, large high-head concrete dams. The ARTeMIS Modal software package [10] was used. A double-ended estimate was constructed for the first singular value of the CSDM, subsequently used to theoretically substantiate the main criterion of the FDD method in the case of ‘single’ frequencies [1]. Furthermore, a new modification of the FDD method for determining damping ratios was developed [2], providing an alternative to existing FDD methods, namely, enhanced (EFDD) and frequency-spatial (FSDD) [5, 6, 9]. With all the evidence presented and the conclusions drawn, it was assumed that all elements of the CDSM could be determined based on the measurement results.

The main goal of this study is to substantiate the feasibility of using such a CDSM where not all of its elements are known to obtain the required dynamic characteristics.

We refer to such a matrix as incomplete. The incompleteness of the matrix is of fundamental importance, since a significant number of measuring points is generally required for the detailed identification of eigenmodes of large and structurally complex structures; it should greatly exceed the possible number of simultaneously used sets of measuring equipment.

Even though introducing incomplete CSDMs into such problems is fairly common, the validity of this approach is yet to be substantiated.

Main aspects of FDD method

The following system of motion equations of an object's point masses is considered as a mathematical model in the FDD method:

$$\mathbf{M}\ddot{\mathbf{y}}(t)+\mathbf{C}\dot{\mathbf{y}}(t)+\mathbf{K}\mathbf{y}(t)=\mathbf{x}(t), \tag{1}$$

where $\mathbf{x}(t)$ are the loads; $\mathbf{y}(t)$ is the response ($y_i(t)$ i = 1, 2, ..., N); **M**, **C, K** are symmetric and real matrices of mass, damping, and stiffness, respectively.

The dimension of **M**,**C, K** is determined by the number of degrees of freedom N and is equal to $N \times N$. The matrix **M** is positive definite, the matrices **C** and **K** are positive semidefinite [3].

Because the eigenmodes $\boldsymbol{\varphi}_m$ are linearly independent, the response $\mathbf{y}(t)$ of the system is uniquely represented as their linear combination:

$$\mathbf{y}(t)=\boldsymbol{\varphi}_1\cdot q_1(t)+\boldsymbol{\varphi}_2\cdot q_2(t)+\ldots=\boldsymbol{\Phi}\,\mathbf{q}(t), \tag{2}$$

where $\boldsymbol{\Phi}$ is a matrix with eigenmode vectors in its columns, $\boldsymbol{\Phi} = [\boldsymbol{\varphi}_1,\boldsymbol{\varphi}_2,\ldots,\boldsymbol{\varphi}_M]$ (M is the number of eigenmodes taken into account in this decomposition); $q_i(t)$ are modal coordinates.

As established in [3], if the external force is assumed to be white noise, and the dissipation is small, then the CSDM (denoted as $\mathbf{G}_y(\omega)$) can be expressed in terms of eigenmodes (modal vectors) as follows:

$$\mathbf{G}_y(\omega)=\sum_{m=1}^{M}\frac{c_m\,\boldsymbol{\varphi}_m\boldsymbol{\varphi}_m^{\mathrm{H}}}{i\omega-\lambda_m}+\frac{c_m\,\boldsymbol{\varphi}_m\boldsymbol{\varphi}_m^{\mathrm{H}}}{-i\omega-\lambda_m^*}=\boldsymbol{\Phi}\cdot\operatorname{diag}\left(\alpha_m(\omega)\right)\cdot\boldsymbol{\Phi}^{\mathrm{H}} \tag{3}$$

where $\boldsymbol{\varphi}_m$ is the eigenmode (modal vector); c_m is a positive coefficient; λ_m is the root of the characteristic equation,

$$\lambda_m=-\gamma_m+i\omega_{dm}, \tag{4}$$

$$\alpha_m(\omega)=\frac{c_m\gamma_m}{(\omega-\omega_{md})^2+\gamma^2_m} \tag{5}$$

The following notations are used in Eqs. (4) and (5):

$$(\gamma_m=\omega_{0m}\varsigma_m), \tag{6}$$

ς_m is the damping ratio, ω_{0m} is the natural frequency without damping, ω_{dm} is the natural frequency with damping.

Let us introduce the matrix notation

$$\mathbf{A}=\operatorname{diag}\left(\sqrt{\alpha_m(\omega)}\right). \tag{7}$$

It was established in [3, 4] that

$$\alpha_m>0. \tag{8}$$

Then, expression (3) for $\mathbf{G}_y(\omega)$ can be written as follows:

$$\mathbf{G}_y(\omega)=\boldsymbol{\Phi}\mathbf{A}^2\boldsymbol{\Phi}^{\mathrm{H}}, \tag{9}$$

or, in terms of eigenvectors,

$$\mathbf{G}_y(\omega)=\sum_{m=1}^{M}\alpha_m\boldsymbol{\varphi}_m\boldsymbol{\varphi}_m^{\mathrm{H}}. \tag{10}$$

For simplcity, below we omit the argument ω for functions depending on it, i.e., the notation is $\mathbf{G}_y$ instead of $\mathbf{G}_y(\omega)$, α_m instead of $\alpha_m(\omega)$, etc.

Modal vectors $\boldsymbol{\varphi}_m$ can be regarded as normalized, since, according to expression (5), the coefficient α_m contains a constant c_m, into which a normalization coefficient can be introduced.

The CSDM can be written in coordinate form:

$$\mathbf{G}_y = \begin{bmatrix} \sum_{m=1}^{M} \alpha_m (\varphi_m^{(1)})^2 & \sum_{m=1}^{M} \alpha_m \varphi_m^{(1)} \varphi_m^{(2)} & & \sum_{m=1}^{M} \alpha_m \varphi_m^{(1)} \varphi_m^{(N)} \\ \sum_{m=1}^{M} \alpha_m \varphi_m^{(2)} \varphi_m^{(1)} & \sum_{m=1}^{M} \alpha_m (\varphi_m^{(2)})^2 & \dots & \sum_{m=1}^{M} \alpha_m \varphi_m^{(2)} \varphi_m^{(N)} \\ \dots & \dots & & \dots \\ \sum_{m=1}^{M} \alpha_m \varphi_m^{(N)} \varphi_m^{(1)} & \sum_{m=1}^{M} \alpha_m \varphi_m^{(N)} \varphi_m^{(2)} & & \sum_{m=1}^{M} \alpha_m (\varphi_m^{(N)})^2 \end{bmatrix}. \quad (11)$$

This matrix has the dimension $N \times N$, where N is the dimension of the modal vectors $\{\varphi_m, m=1,...,M\}$, M is the number of modal vectors taken into account in the solution of the problem.

The algorithm of the FDD method is that the results of dynamic tests of an object in a certain frequency range are used to construct first a sequence of CSDM of the measured signals (based on a sequence of frequencies), and then a function of the first singular value of the frequency (based on the singular-value decomposition of these matrices) [3–5].

Let us give the estimates of the first singular value of the CSDM constructed in [1]:

$$\max_{1 \le i \le M} \alpha_i \le \sigma_1 \le \max_{1 \le i \le M} \sum_{j=1}^{M} \alpha_j \left| (\varphi_i \varphi_j) \right|. \quad (12)$$

Based on these estimates, the main criterion of the FDD method was proved in [1]: the first singular value of the CSDM has local maxima near natural (modal) frequencies (referred to as the ‘criterion’ from now on). The proof was carried out provided that all the elements of the matrix $\mathbf{G}_y$ (CSDM) are known. However, as noted above, the results of dynamic tests of large structures, such as high-head concrete dams, used to identify dynamic characteristics, make it possible to determine only a small part of CSDM values.

Application of FDD method to incomplete CSDMs

Before substantiating the use of incomplete CSDMs, let us briefly describe the measurement procedure. The response vector $\mathbf{y}(t)$ is found during dynamic tests (DT) (see Eqs. (1) and (2)); for this purpose, derivatives of the signals y_i(t) are measured for some period at pre-determined points of the object under study (accelerations are typically measured, less often velocities).

Fig. 1 shows an example of a measurement circuit on the crest and in the galleries of a dam (a gallery is a longitudinal channel within a hydraulic structure (dam, etc.)). Evidently, a very large number of measuring points are required for detailed identification of eigenmodes in large structures. Apparently, it is technically impossible to install a sensor (accelerometer or velocimeter) at each measuring point for synchronous measurements. As a result, the dynamic testing technique provides two types of measurements:

stationary, performed continuously at one or several reference points;

short-term, ‘mobile’ (5–10 minutes long), performed sequentially at selected points of the examined structure by the same sensors.

The position of the reference measuring points should not change until the end of the tests.

Without loss of generality, we assume that only one of the three components of vibration acceleration is measured, the longitudinal one, providing the largest contribution to the configurations of the lower eigenmodes in high-head dams. The cross-correlation function $R_{ij}(t)$ and the cross-spectral density $S_{ij}(\omega)$, which is the Fourier transform of the function $R_{ij}(t)$, can be calculated for any pair of signals with numbers i and j measured simultaneously. The frequency functions $S_{ij}(\omega)$ are (i,j)-elements of the CSDM with dimension $N \times N$, where N is the number of measurement points. Note that since the function $S_{ij}(\omega)$ is calculated by the measurement data using the fast Fourier transform over a discrete frequency range $\omega_{p}, _{p=1..P}$, a sequence of the matrix $\mathbf{G}_y(\omega_p)$ with the elements $S_{ij}(\omega_p)$ is obtained from the DT data.

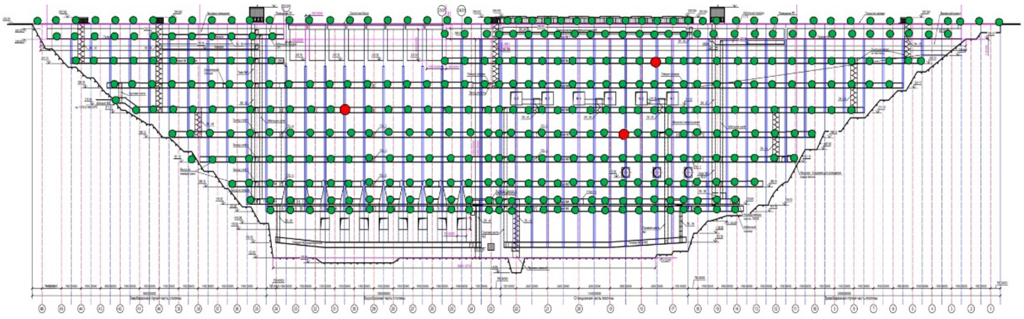

Fig. 1. Example of measuring circuit for dynamic tests on high-head concrete dam
Reference measuring points (red) and regular grid of consecutive measurements (green dots) are shown

If the signals were measured at all measuring points simultaneously, then all elements of the matrix $\mathbf{G}_y(\omega_p)$ of dimension $N \times N$ $(\omega_p, _{p=1..P})$ would be known (we refer to such matrices as complete CSDM from now on).

The criterion was proved precisely for such matrices in [1]. However, in accordance with the DT technique described above, the cross-correlation function $R_{ij}(t)$ (and consequently $S_{ij}(\omega)$) can be calculated only for such a pair of measurements, one of which was obtained at a stationary measuring point, or in the case when $i = j$, i.e., auto-spectral density.

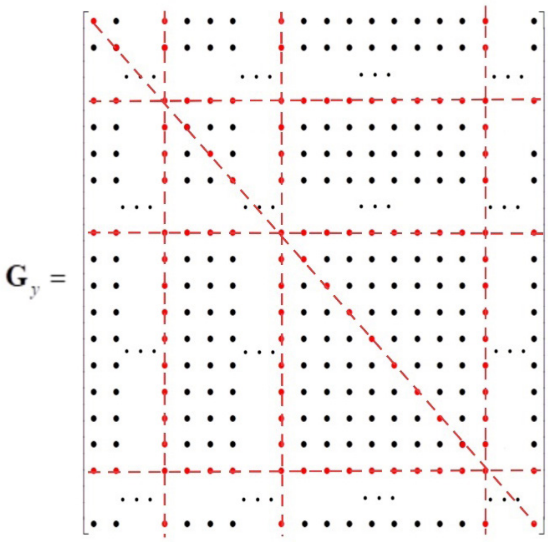

Fig. 2. Scheme of incomplete CSDM, showing an example for the arrangement of the known CSDM elements (red dots connected by a dashed line)

Therefore, as a result of the DT, we can determine only a small part of the values of the CSDM. These are the elements located on the main diagonal (auto-spectral densities of all measured signals), as well as elements in several columns and, accordingly, rows of the matrix, whose numbers correspond to the numbers of the reference measuring points.

Fig. 2 schematically shows how the known CSDM elements can be arranged.

Let us compose matrices of the maximum possible dimension from the known elements of the matrix $\mathbf{G}_y$ (for each frequency ω_p). We denote sequences of such matrices $\mathbf{G}_y^r(\omega_p, _{p=1..P})$ (for brevity, the notation $\mathbf{G}_y^r$ is used).

We denote the numbers of the reference points as $i_1, i_2, ..., i_K$, while

$$i_1 < i_2 < \ldots < i_K,$$

where K is the number of reference points.

Let these numbers form a set M of the form

$$\mathrm{M} = \{i_1, i_2, \ldots, i_K\},$$

and the numbers of the measurement points form a set $\tilde{\mathrm{R}}$ of the form

$$\tilde{\mathrm{R}} = 1, 2, ..., N, \quad (12)$$

where N is the total number of measuring points.

Next, let j be any number from 1 to N that does not coincide with the reference point number, i.e., $j \subset \tilde{\mathrm{R}} / \mathrm{M}$.

Consider sets of the following form:

$$\mathrm{M}_j = \{i_1, i_2, \ldots, j, \ldots, i_K\}, \text{ where } i_1 < i_2 < \ldots < j < \ldots < i_K,$$

here, if $j < i_1$, then $\mathrm{M}j = \{j, i_1, i_2, ..., i_K\}$, or $j > i_K$, then $\mathrm{M}_j = \{i_1, i_2, ..., i_K, ..., j\}$.

Evidently, the number of such sets is $N - K$.

We choose the smaller matrices $\mathbf{G}_y^r$, r=1..N−K from the matrix $\mathbf{G}_y$, so that they consist of elements lying at the intersection of rows and columns with numbers belonging to sets M_j. As a result, $N - K$ square matrices of dimension $(i_K + 1) \times (i_K + 1)$ can be selected from the initial matrix. These are matrices of the following form:

$$\tilde{\mathbf{G}}_y^r = \begin{bmatrix} \mathbf{G}_y(i_1,i_1) & & \mathbf{G}_y(i_1,j) & & \mathbf{G}_y(i_1,i_K) \\ \dots & & \dots & & \dots \\ \mathbf{G}_y(j,i_1) & \dots & \mathbf{G}_y(j,j) & \dots & \mathbf{G}_y(j,i_K) \\ \dots & & \dots & & \dots \\ \mathbf{G}_y(i_K,i_1) & & \mathbf{G}_y(i_K,j) & & \mathbf{G}_y(i_K,i_K) \end{bmatrix}. \quad (13)$$

Note that the cases $j < i_1$ and $j > i_K$ are also considered.

Thus, each of the matrices $\mathbf{G}_y^r$ contains all the elements that lie at the intersection of the rows and columns corresponding to the numbers of the reference points and at the intersection of the row (column) numbered j ($j \subset \tilde{\mathrm{R}} / \mathrm{M}$) with the columns (rows) with the numbers of the reference points.

The matrix $\mathbf{G}_y^r$ can be written as follows in coordinate form:

$$\mathbf{G}_y^r = \begin{bmatrix} \sum_{m=1}^{M} \alpha_m \varphi_m^{(i_1)} \varphi_m^{(i_1)} & & \sum_{m=1}^{M} \alpha_m \varphi_m^{(i_1)} \varphi_m^{(j)} & & \sum_{m=1}^{M} \alpha_m \varphi_m^{(i_1)} \varphi_m^{(i_K)} \\ \dots & & \dots & & \dots \\ \sum_{m=1}^{M} \alpha_m \varphi_m^{(j)} \varphi_m^{(i_1)} & \dots & \sum_{m=1}^{M} \alpha_m \varphi_m^{(j)} \varphi_m^{(j)} & \dots & \sum_{m=1}^{M} \alpha_m \varphi_m^{(j)} \varphi_m^{(i_K)} \\ \dots & & \dots & & \dots \\ \sum_{m=1}^{M} \alpha_m \varphi_m^{(i_K)} \varphi_m^{(i_1)} & & \sum_{m=1}^{M} \alpha_m \varphi_m^{(i_K)} \varphi_m^{(j)} & & \sum_{m=1}^{M} \alpha_m \varphi_m^{(i_K)} \varphi_m^{(i_K)} \end{bmatrix}. \quad (14)$$

Let us introduce the following notation:

$$\boldsymbol{\varphi}_m^r = \left[\varphi_m^{(i_1)}, \dots, \varphi_m^{(j)}, \dots, \varphi_m^{(i_K)} \right]. \quad (15)$$

The component of the modal vector $\varphi_m^{(j)}$ can be located at any position depending on the number j (for example, at the very beginning or end of the vector $\boldsymbol{\varphi}_m^r$). Notably, in this case, $\boldsymbol{\varphi}_m^r$ is a vector consisting of several components of the modal vector. Define the vector $\tilde{\boldsymbol{\varphi}}_m^r$:

$$\tilde{\boldsymbol{\varphi}}_m^r = \left[\tilde{\varphi}_m^{r\,(1)}, \tilde{\varphi}_m^{r\,(2)}, \dots, \tilde{\varphi}_m^{r\,(K+1)} \right], \quad (16)$$

where $\tilde{\varphi}_m^{r\,(1)} = \varphi_m^{(i_1)}, \dots, \tilde{\varphi}_m^{r\,(K)} = \varphi_m^{(i_K)}$.

Then the matrices $\tilde{\mathbf{G}}_y^r$ can be rewritten as follows:

$$\mathbf{G}_y^r = \begin{bmatrix} \sum_{m=1}^{M} \alpha_m \tilde{\varphi}_m^{r\,(1)} \tilde{\varphi}_m^{r\,(1)} & & \sum_{m=1}^{M} \alpha_m \tilde{\varphi}_m^{r\,(1)} \tilde{\varphi}_m^{r\,(s)} & & \sum_{m=1}^{M} \alpha_m \tilde{\varphi}_m^{r\,(1)} \tilde{\varphi}_m^{r\,(K+1)} \\ \dots & & \dots & & \dots \\ \sum_{m=1}^{M} \alpha_m \tilde{\varphi}_m^{r\,(s)} \tilde{\varphi}_m^{r\,(1)} & \dots & \sum_{m=1}^{M} \alpha_m \tilde{\varphi}_m^{r\,(s)} \tilde{\varphi}_m^{r\,(s)} & \dots & \sum_{m=1}^{M} \alpha_m \tilde{\varphi}_m^{r\,(s)} \tilde{\varphi}_m^{r\,(K+1)} \\ \dots & & \dots & & \dots \\ \sum_{m=1}^{M} \alpha_m \tilde{\varphi}_m^{r\,(K+1)} \tilde{\varphi}_m^{r\,(1)} & & \sum_{m=1}^{M} \alpha_m \tilde{\varphi}_m^{r\,(K+1)} \tilde{\varphi}_m^{r\,(1)} & & \sum_{m=1}^{M} \alpha_m \tilde{\varphi}_m^{r\,(K+1)} \tilde{\varphi}_m^{r\,(K+1)} \end{bmatrix}. \quad (17)$$

Comparison of the matrix $\mathbf{G}_y^r$ (17) with the complete matrix $\mathbf{G}_y$ (11) shows that they have exactly the same structure. Notably, to obtain estimates of the first singular value by the method described in our article [1], the only requirement for the vectors $\boldsymbol{\varphi}_m$ is their normalization. In the case of a complete matrix, this requirement was fulfilled, since, in accordance with expression (5), the function $\alpha_m(\omega)$ contains some unknown constant c_m into which a normalization coefficient can be introduced. Indeed, to prove the criterion in [1], we used the fact that the functions $\alpha_m(\omega)$ have maxima at natural frequencies and the value of the constant c_m is of no importance (it only matters that it is positive). Therefore, we can consider the eigenmode vectors in Eq. (10) to be normalized because c_m can be multiplied by the normalization coefficient necessary for normalization of $\boldsymbol{\varphi}_m$.

In this paper, we choose an incomplete matrix $\mathbf{G}_y^r$ and can similarly consider the vectors $\tilde{\boldsymbol{\varphi}}_m^r$ to be normalized based on the constants c_m (see Eq. (5)), each of which can also be multiplied by the normalization coefficient needed to normalize the vector $\tilde{\boldsymbol{\varphi}}_m^r$ and denote it c_m^r. Then Eq. (5) is rewritten as follows:

$$\alpha_m^r(\omega) = \frac{c_m^r \gamma_m}{(\omega - \omega_{md})^2 + \gamma_m^2}. \tag{18}$$

Therefore, to derive estimates of the first singular value of the matrix $\mathbf{G}_y^r$, we can apply the method described in detail and substantiated in [1] for the complete matrix $\mathbf{G}_y$. It was also established in [1] that the matrix $\mathbf{G}_y$ is Hermitian and positive semidefinite, so its singular values and eigenvalues coincide. To obtain estimates of the first singular number, auxiliary matrices having the same spectral radius with the matrix $\mathbf{G}_y$ were used in [1]. The first such matrix $\mathbf{K}$ is the Gram matrix [14], and in the case of a complete matrix it is constructed from the vectors $\sqrt{\alpha_m}\boldsymbol{\varphi}_m$. Since $\mathbf{G}_y$ is described by Eq. (9), the corresponding Gram matrix is written as

$$\mathbf{K} = \mathbf{A}(\boldsymbol{\Phi}^{\mathrm{H}}\boldsymbol{\Phi})\mathbf{A}. \tag{19}$$

It was proved in [1] that the lower bound of the first singular number constructed for the matrix $\mathbf{K}$ is also the corresponding bound for the matrix $\mathbf{G}_y$.

The matrix $\mathbf{T}$ is defined by the expression

$$\mathbf{T} = (\boldsymbol{\Phi}^{\mathrm{H}}\boldsymbol{\Phi})\mathbf{A}^2. \tag{20}$$

Taking into account the complete identity of (11) and (17), we can replace the modal vectors $\boldsymbol{\varphi}_m$ with $\tilde{\boldsymbol{\varphi}}_m^r$, and the coefficients α_m with α_m^r, obtaining a double-ended estimate of the first singular number σ_1^r of the matrix $\mathbf{G}_y^r$ (r = 1, 2, ..., $N - K$):

$$\max_{1\le i\le M} \alpha_i^r \le \sigma_1^r \le \max_{1\le i\le M} \sum_{j=1}^{M} \alpha_j^r \left|(\tilde{\boldsymbol{\varphi}}_i^r \tilde{\boldsymbol{\varphi}}_j^r)\right| \le \sum_{k=1}^{M} \alpha_k^r = \mathrm{Tr}(\widetilde{\mathbf{G}}_y^r). \tag{21}$$

Since the functions $\alpha_i^r(\omega)$ differ from $\alpha_i(\omega)$ only by constant coefficients, then, just as in the case of a complete matrix for a ‘single’ natural frequency, a double-ended estimate$\sigma_1^r(\omega)$ ensures that the criterion is fulfilled, i.e., the natural frequencies are near the abscissa of the maxima of functions $\sigma_1^r(\omega)$. Note that estimates (21) hold true over the entire frequency range.

Thus, to determine several lower natural frequencies of a structure by the FDD method and the attenuation parameters by the method described in [2], in the ideal case, i.e., satisfying all conditions for using the FDD method (namely, external forces are stationary and have a spectrum close to ‘white noise’ in a significant frequency range; attenuation is low; measuring points are not located near the nodal lines of the eigenmodes; the measurement length ensures noise suppression, etc., information about the values of two elements on the main diagonal and two elements lying at the intersection of the row and column corresponding to these elements is sufficient).

It was proved in [1] that in the case of an arbitrary dimension of the CSDM, single frequencies are identified as arguments of maxima of the function of the first singular number. On the other hand, the present study shows that it is possible to choose a matrix from an incomplete matrix in such a way that we arrive at the case described above.

Since the real conditions in which DT are preformed differ from ideal ones, several reference points are typically used, so consequently we obtain $N - K$ sequences of matrices $\mathbf{G}_y^r(\omega_p)$, for each of which we can construct the function $\sigma_1^r(\omega_p)$, and then determine the lowest natural frequencies by the algorithm based on the criterion (described in detail in [1]).

Up to 10–14 natural frequencies can be identified using the described approach for high-head concrete dams. The final values of the natural frequencies are obtained by averaging the results of applying the criterion over all $N - K$ sets. In addition to simple averaging, more complex integral statistics can be used, as well as additional indicators ranking the matrices $\mathbf{G}_y^r(\omega_p)$ by quality of the measured signals and effect of measurement noise.

The method for determining the eigenmodes in the case of incomplete CSDM also requires some modification. It was shown in [1, 3] for complete CSDM that the eigenmode corresponding to a single frequency with the number m can be well estimated by the first singular vector of the matrix $\mathbf{G}_y(\omega_m)$. Since it is Hermitian and positive semidefinite, its first singular vector coincides with its eigenvector corresponding to the maximum eigenvalue. Moreover, since the rank of the matrix $\mathbf{G}_y(\omega_m)$ for a single natural frequency (ω_m) can be considered singular, this eigenvector coincides with any column vector (row vector) of the matrix $\mathbf{G}_y(\omega_m)$ with an accuracy of a coefficient (see [1, 3]). Therefore, in the case of complete CSDM, the eigenmode practically coincides with any column (row) of this matrix calculated at the corresponding natural frequency. In particular, any column (row) of the matrix $\mathbf{G}_y(\omega_m)$ for the natural frequency (ω_m) is the eigenmode corresponding to this frequency.

This method for determining the eigenmode is generally unsuitable for incomplete matrices. This is because the dynamic test method described above assumes that measurements at ‘mobile’ points are not performed simultaneously, and, therefore, can be performed at different dynamic loads. For example, sequential measurements at points of the HS are usually performed under different operating modes of the turbine. The dynamic characteristics of linear dynamic system (1) do not depend on loads, so estimates of natural frequencies using $\mathbf{G}_y^r(\omega_p)$ can be averaged, and the vectors $\tilde{\boldsymbol{\varphi}}_m^r$ give estimates of values for several components of eigenmodes.

The following technique is proposed in [3, 11] to reconstruct the complete eigenmode vectors. Let the matrix $\mathbf{G}_y^r(\omega_m)$ be calculated at any natural frequency ω_m (the argument will be omitted for brevity). The part of the eigenmode vector (with components whose numbers are $i_1, i_2, \ldots, i_K$) is determined from a data set common to all matrices $\mathbf{G}_y^r$. We denote this part as $\bar{\mathbf{b}}^r$. The number of components of the vector $\bar{\mathbf{b}}^r$ corresponds to the number of reference points; the remaining $\bar{\bar{\mathbf{b}}}^r$ components are determined from the corresponding matrix $\mathbf{G}_y^r$. Then the part of the eigenmode vector for the degrees of freedom determined by the matrix $\mathbf{G}_y^r$ can be written as

$$\boldsymbol{\varphi}^r = \begin{bmatrix} \bar{\mathrm{b}}^r \\ \bar{\bar{\mathrm{b}}}^r \end{bmatrix}. \tag{22}$$

Evidently, the vectors $\bar{\mathbf{b}}^r$ must be proportional for any number r. If we take the vector $\bar{\mathbf{b}}^1$ as a basis, then the scaling between them can be determined as follows:

$$\bar{\mathrm{b}}^1 = a_{1i}\bar{\mathrm{b}}^i. \tag{23}$$

The complete eigenvector takes the following form:

$$\boldsymbol{\varphi} = \begin{bmatrix} \bar{\mathrm{b}}^1 \\ \bar{\bar{\mathrm{b}}}^1 \\ a_{12}\bar{\bar{\mathrm{b}}}^2 \\ \ldots \\ a_{1K}\bar{\bar{\mathrm{b}}}^K \end{bmatrix}. \tag{24}$$

Since the vectors $\bar{\mathbf{b}}^r$ are determined with some error, in the case when the dimension of the vectors $\bar{\mathbf{b}}^i$ exceeds unity, it is advisable to use the OLS method to find the scaling factors.

The eigenmodes of a structure are generally obtained based on a number of measurement points allowing to construct the configurations of these eigenmodes. As a rule, the higher the sequence number of the dynamic characteristics, the denser the measuring grid should be. In other words, the dimension of the complete CSDM is dictated by the number of degrees of freedom in the mathematical model of the object under study. If an object with an infinite number of degrees of freedom is simulated by a linear dynamical system with N equations, the complete CSDM has the dimension $N \times N$, and in order to apply the FDD method in this matrix, the elements of the main diagonal and the elements of at least one row (and, due to the properties of the CSDM, respectively, the column) must be known.

Conclusion

The paper provides theoretical substantiation for the algorithm for determining the dynamic characteristics, i.e., natural frequencies and eigenmodes, found by vibration tests with sequential asynchronous measurements. Several reference points where measurements are taken continuously are sufficient for applying the FDD algorithm in this case. Only a certain part of the elements of the cross-spectral density matrix (CSDM) of signals measured at various points of the structure can be calculated in these tests. A detailed description of the algorithm for constructing matrices consisting of known elements of the initial incomplete CSDM is presented, whose singular-value decomposition makes it possible to determine the natural frequencies of the structure under study.

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