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# OSCILLATIONS UNDER A NONLINEAR PARAMETRIC ACTION AND COMBINATIONS OF DELAYS

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**Abstract.** The paper considers oscillations under nonlinear parametric action and combinations of delays in elasticity and damping. The model for the study is a rod with a spring, which is driven by an energy source of limited power. To solve nonlinear differential equations of motion of the system, the method of direct linearization of nonlinearity has been used. Equations were obtained for determining the nonstationary and stationary values of the amplitude and phase of oscillations, the speed of the energy source. Based on the Routh–Hurwitz criteria, the conditions for the stability of stationary motion modes were derived. To obtain information about the combined effect of delays on the dynamics of oscillations, the calculations were carried out for their various values, linear and nonlinear elastic forces. The graphs constructed based on the calculation results clearly show the combined effect of various delay values on the amplitude-frequency curves. The delays measure the amplitude curve, shift it to the right-left, up-down, and affect the stability of the oscillations.

**Keywords:** oscillation, model, nonlinearity, method, parametric excitation, delay, elasticity, damping

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# КОЛЕБАНИЯ ПРИ НЕЛИНЕЙНОМ ПАРАМЕТРИЧЕСКОМ ВОЗДЕЙСТВИИ И КОМБИНАЦИИ ЗАПАЗДЫВАНИЙ

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**Аннотация.** В работе рассмотрены колебания при нелинейном параметрическом воздействии и комбинации запаздываний в упругости и демпфировании. Моделью является стержень с пружиной, приводимый в движение источником энергии ограниченной мощности. Для решения нелинейных дифференциальных уравнений движения системы использован метод прямой линеаризации нелинейности. Получены уравнения для определения нестационарных и стационарных значений амплитуды и фазы колебаний, скорости источника энергии. На основе критериев Рауса – Гурвица выведены условия устойчивости стационарных режимов движения. Проведены расчеты амплитудно-частотных характеристик при различных значениях параметров, линейной и нелинейной силах упругости. Соответствующие графики наглядно представляют совместное влияние различных значений запаздываний на амплитудно-частотные кривые. Показано, что запаздывания изменяют амплитудные кривые, существенно влияя на устойчивость колебаний.

**Ключевые слова:** колебания, модель, нелинейность, метод, параметрическое воздействие, запаздывание, упругость, демпфирование

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## Introduction

All phenomena in nature (the Universe) are cyclical, oscillatory motion occurs in all types of systems (physical, biological, technical, etc.) [1]. Excitation of oscillations can be caused by various reasons, including the presence of a delay in many systems [2, 3, etc.]. The appearance of delays in mechanical systems can be caused by the elasticity of materials and internal friction in them. Many studies considered time-delay systems [3—17, etc.], however, they did not take into account the properties of the source supplying energy to the system. Real physical systems function using some kind of energy source with limited power. This is scarcely addressed in the literature. Energy consumption issues, as well as related environmental and climate change issues, have now become particularly important.

In most cases, time-delay systems are analyzed based on nonlinear differential equations with a deviating argument. These equations are solved by various methods of nonlinear mechanics [18—20, etc.], characterized by high labor and time costs. The direct linearization method (DLM), described in [21—23] and other works, does not have these costs, which provides an advantage over known methods of nonlinear mechanics. Its essential properties are also simplicity and the possibility of obtaining finite ratios regardless of the degree of nonlinearity, which makes it easy to use it in practical calculations.

Many systems (pendulum with a vibrating pivot, shaft, driveshaft, gear train, railway bridge, etc.) experience parametric oscillations, which can be caused by both linear and nonlinear excitations. Parametric oscillations under linear and nonlinear (quadratic) excitations were considered in monograph [24].

The goal of this study is to analyze parametric oscillations taking into account the properties of the energy source, nonlinear parametric action (cubic) and the presence of delays in elasticity and friction.

## Equations of the system and solutions

Let us take as a basis the model and equations (formulated assuming that oscillations of the rod taking the form of the first eigenmode of free bending oscillations), where the dynamics of the system is supported by a limited-power motor (Fig. 1) [25]. Taking into account the nonlinearity of parametric excitation as well as delays in elasticity and friction, we obtain the following equations of motion:

$$\begin{gathered} \ddot{y} + \beta_1 \dot{y} + \omega^2 y + b y^3 \sin\varphi = -m^{-1} f(y) - k_\eta \dot{y}_\eta - c_\tau y_\tau, \\ J\ddot{\varphi} = M(\dot{\varphi}) - 0.5 c_2 y^2 \cos\varphi - 0.5 c_3 \sin 2\varphi - c_4 \cos\varphi, \end{gathered} \quad (1)$$

where $\omega^2 = \dfrac{c}{m}$, $c = \dfrac{\pi^4 E I_x}{2l^3}\left(1 - \dfrac{P_0}{P_1}\right)$, $b = \dfrac{c_2}{m}$, $c_2 = -\dfrac{\pi^2 r_1 c_1}{2l}$, $m = \dfrac{\rho l}{2}$, $\beta_1 = \dfrac{\beta}{m}$, $P_0 = f_0 c_1$,

$P_1 = \dfrac{\pi^2 E I_x}{l^2}$, $c_3 = c_1 r_1^2$, $c_4 = f_0 r_1 c_1$.

The quantities $\omega$, $c$, $m$, $\beta$, $b$, $c_2$, $c_3$, $c_4$, $P_0$, $P_1$ in Eqs. (1) are constant; $c_1$, $\beta$ are the spring constant and the resistance coefficient; $\rho$ is the mass per unit length of the rod; $EI_x$ is the bending stiffness of the rod along the $y$ axis; $f_0$ is the precompression on the spring; $f(y)$ is the nonlinear component of elasticity; $c_\tau = const$, $k_\eta = const$ , $\dot{y}_\tau = y(t{-}\tau)$, $\dot{y}_\eta = y(t{-}\eta)$,$\tau = const$ and $\eta = const$ – are the delays; $J$ is the moment of inertia of the motor rotor rotating the crank of radius $r_1$ connected to the spring; $M(\dot{\varphi})$ is the driving torque of the motor (taking into account the resistance forces); $\dot{\varphi}$ is the rotational speed of the motor.

In practice, representation of nonlinearity by means of a polynomial function has become widespread. Let us adopt it as the nonlinear component of the elastic force $f(y)$ in the form

$$f(y) = \sum_s \gamma_s y^s,$$

where $\gamma_s = const$, $s = 2, 3 \ldots$ .

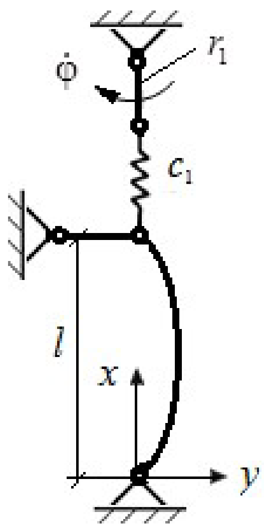


Fig. 1. Model of oscillatory system:
$r_1$ is the crank radius; $l$ is the geometric size; $c_1$ is the spring constant; $\varphi$ is the rotational speed of the motor

$$f_*(y) = B_f + c_f y.$$

Here $B_f$, $c_f$ are the linearization coefficients defined by the expressions

$$B_f = \sum_s N_s \gamma_s a^s,\ s = 2,4,6,\ \ldots\ (s \text{ is even}),$$
$$c_f = \sum_s \bar{N}_s \gamma_s a^{s-1},\ s = 3,5,7,\ \ldots\ (s \text{ is odd}), \quad (2)$$

where $a = \max|y|$; $N_s = (2r + 1)/(2r + 1 + s)$, $r$ is the linearization accuracy parameter, whose selection interval is unlimited but sufficient within 0–2.

Taking into account expressions (2), Eqs. (1) take the form

$$\ddot{y} + \beta_1 \dot{y} + \omega^2 y + b y^3 \sin\varphi = -m^{-1}(B_f + c_f y) - k_\eta \dot{y}_\eta - c_\tau y_\tau, \quad (3)$$

$$J\ddot{\varphi} = M(\dot{\varphi}) - 0.5 c_2 y^2 \cos\varphi - 0.5 c_3 \sin 2\varphi - c_4 \cos\varphi.$$

To solve Eqs. (3), we apply the DLM and the procedure presented in [23] and other studies.

Using the functions

$$y = a\cos\psi,\ \ y_\tau = a\cos(\psi - p\tau),\ \ \dot{y}_\eta = -ap\sin(\psi - p\eta),\ \ \dot{\varphi} = \Omega,\ \ \psi = pt + \xi,\ \ p = \Omega/2,$$

we obtain the following equations for nonstationary motion:

$$\frac{da}{dt} = -\frac{a}{2}\left(\beta_1 + k_\eta\cos p\eta - 2c_\tau\Omega^{-1}\sin p\tau\right) + \frac{ba^3}{4\Omega}\cos 2\xi,$$

$$\frac{d\xi}{dt} = \frac{4\omega^2 - \Omega^2}{4\Omega} + \frac{c_f}{m\Omega} + \frac{1}{2}k_\eta\sin p\eta + \frac{c_\tau}{\Omega}\cos p\tau - \frac{ba^2}{2\Omega}\sin 2\xi, \qquad (4)$$

$$\frac{d\Omega}{dt} = \frac{1}{J}\left[M(\Omega) - \frac{c_2 a^2}{8}\right].$$

The conditions $\dot{a} = 0,\ \dot{\xi} = 0,\ \dot{\Omega} = 0$, provide the following relations for stationary motion

$$4m^2A^2 + D^2 = 4m^2b^2a^4,$$

$$tg2\xi = -D/2mA, \qquad (5)$$

$$M(\Omega) - S(a) = 0,$$

where $A = 2\Omega(\beta_1 + k_\eta\cos p\eta) - 4c_\tau\sin p\tau,$

$D = m(4\omega^2 - \Omega^2) + 4c_f + 2m\Omega k_\eta\sin p\eta + 4mc_\tau\cos p\tau,\ \ S(a) = c_2a^2/8.$

The expression $S(a)$ represents the load on the energy source from the oscillatory system. The intersection points of the curves $M(\Omega)$ and $S(a)$ determine the speeds $\Omega$.

**Stability conditions**

Stationary motions need to be analyzed for stability. Composing the equations in variations for Eqs. (4) and using the Routh–Hurwitz criteria, we obtain the conditions for the stability of σατιουαρψ oscillations:

$$D_1 > 0,\ D_3 > 0,\ D_1D_2 - D_3 > 0, \qquad (6)$$

where $D_1 = -(b_{11} + b_{22} + b_{33}),\ D_2 = b_{11}b_{33} + b_{11}b_{22} + b_{22}b_{33} - b_{23}b_{32} - b_{12}b_{21} - b_{13}b_{31},$

$D_3 = b_{11}b_{23}b_{32} + b_{12}b_{21}b_{33} - b_{11}b_{22}b_{33} - b_{12}b_{23}b_{31} - b_{13}b_{21}b_{32},$

$b_{11} = \frac{1}{J}Q,\ b_{12} = -\frac{c_2a}{4J},\ b_{13} = 0,\ b_{21} = -\frac{a}{\Omega^2}c_\tau\sin p\tau - \frac{ba^3}{4\Omega^2}\cos 2\xi,$

$b_{22} = -\frac{1}{2}\left(\beta_1 + k_\eta\cos p\eta - 2c_\tau\Omega^{-1}\sin p\tau\right) + \frac{3ba^2}{4\Omega}\cos 2\xi,\ b_{23} = -\frac{ba^3}{2\Omega}\sin 2\xi,$

$b_{31} = -0,25 - \frac{\omega^2}{\Omega^2} - \frac{c_f}{m\Omega^2} - \frac{c_\tau}{\Omega^2}\cos p\tau + \frac{ba^2}{2\Omega^2}\sin 2\xi,\ b_{32} = \frac{1}{m\Omega}\frac{\partial c_f}{\partial a} - \frac{ba}{\Omega}\sin 2\xi,$

$b_{33} = -\frac{ba^2}{\Omega}\cos 2\xi,\ Q = \frac{d}{d\Omega}M(\Omega).$

The slope of the energy source characteristic $Q = dM/d\Omega$ makes it possible to determine the regions where the oscillations are stable or unstable.

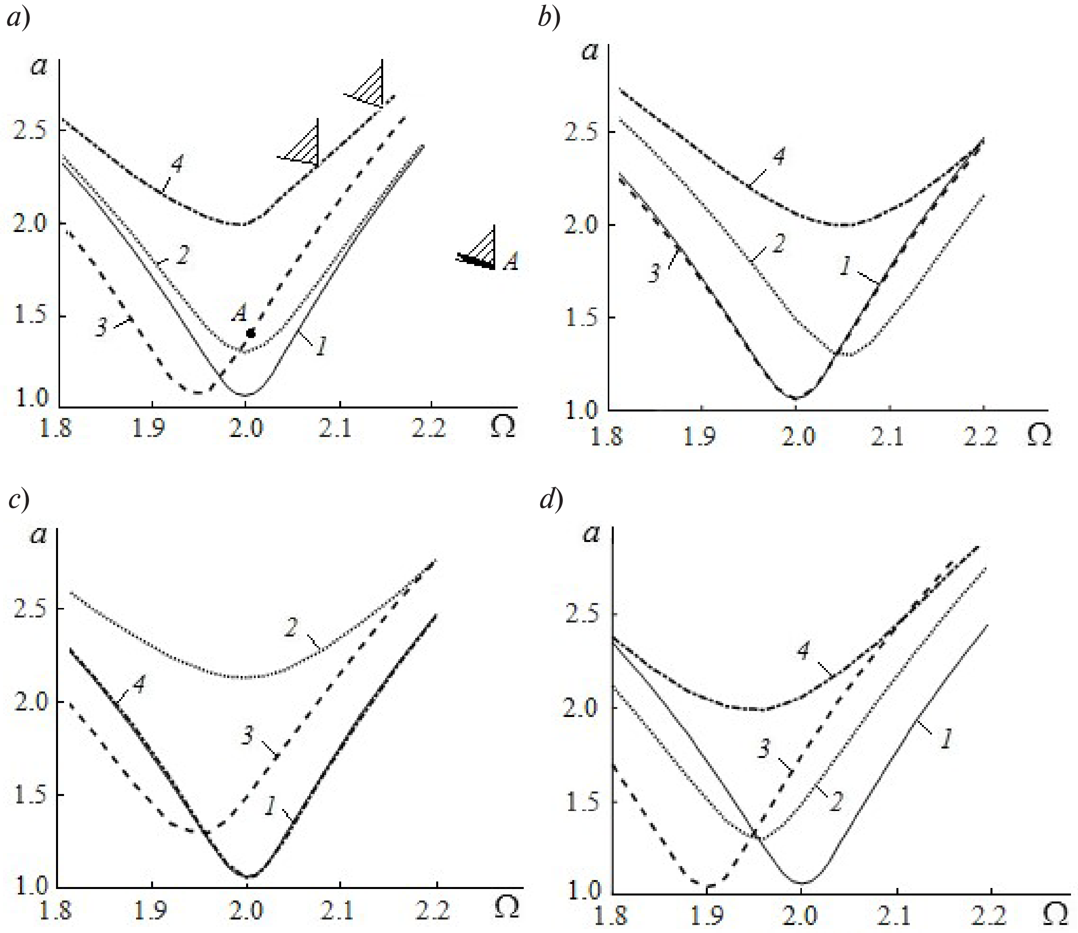


Fig. 2. Amplitude–frequency curves for the case of linear elastic force; $\gamma_3 = 0$; the values of the parameters $p\eta$ and $p\tau$ are varied. For comparison, the case with the absence of delays is given in all graphs ($k_\eta = 0$, $c_\tau = 0$, curves *1*).
The shaded sectors for the slope $Q$ of the energy source characteristic (at point $A$ and others) correspond to stable oscillations. Parameter values: $p\eta = 0$ (*a*), $\pi/2$ (*b*), $\pi$ (*c*), $3\pi/2$ (*d*); $p_\tau = \pi/2$ (curves *2*), $\pi$ (curves *3*), $3\pi/2$ (curves *4*)

**Calculations performed and main results**

Calculations were carried out to obtain information on the effect of nonlinear parametric effects and delays on the dynamics of oscillations. The nonlinear component of the elastic force was taken as

$$f(y) = \gamma_3 y^3,\ \gamma_3 = \pm 0.2\ \text{kgf}\cdot\text{cm}^{-3},$$

and the other parameters had the following values:

$$\omega = 1\ s^{-1},\ m = 1\ \text{kgf}\cdot\text{s}^2\cdot\text{cm}^{-1},\ c_2 = 0.07\ \text{kgf}\cdot\text{cm}^{-1},$$

$$\beta = 0.02\ \text{kgf}\cdot\text{s}\cdot\text{cm}^{-1},\ k_\eta = 0.05\ \text{kgf}\cdot\text{s}\cdot\text{cm}^{-1},\ c_\tau = 0.05\ \text{kgf}\cdot\text{cm}^{-1}.$$

The following values were accepted for delays:

$$p\eta = 0,\ \pi/2,\ \pi,\ ;\ \ p\tau = 0,\ \pi/2,\ \pi, 3\pi/2.$$

The linearization coefficient $\bar{N}_3 = 3/4$, which corresponds to the linearization accuracy parameter $r = 1.5$.

All results calculated by the DLM completely coincide with those calculated by the widely used Bogolyubov–Mitropolsky asymptotic averaging method [18], as the number 3/4 is obtained using both methods.

Figs. 2–4 shows the amplitude–frequency curves $a(\Omega)$ for linear and nonlinear elastic forces (the quantities in the graphs are normalized). All graphs show different parameter values, and the solid curve *1*, which is given for comparison, corresponds to the absence of delays ($k_\eta = 0$, $c_\tau = 0$). Criteria (6) are satisfied within the shaded sectors (see Fig. 2) for the slope $Q$ of the energy source characteristic, and stable fluctuations occur only in fairly narrow frequency ranges at

$$\gamma_3 = 0,\ k_\eta = 0,\ p\tau = \pi \text{ and } p\tau = 3\pi/2.$$

These sectors should be shown on the load curve $S(a)$ but for brevity they are shown instead on the amplitude–frequency curves. There is no stability in the cases when $\gamma_3 = \pm 0.2$ in the entire range of resonant frequencies for all delays considered.

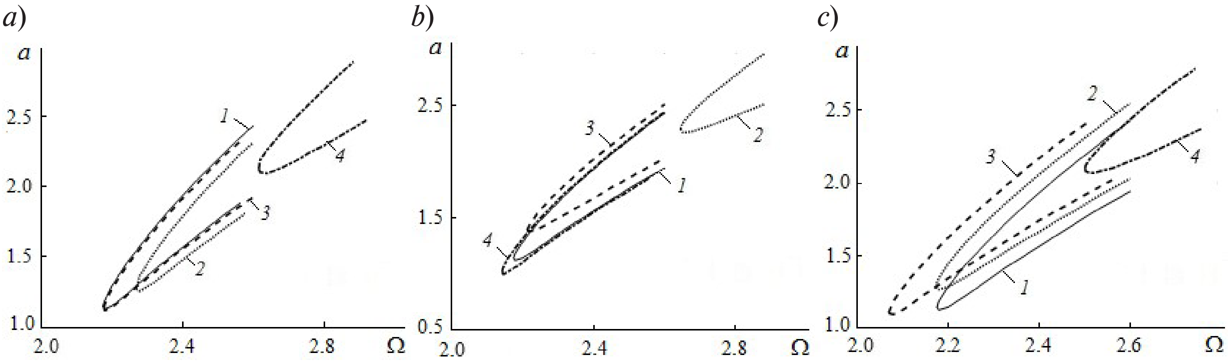


Fig. 3. Amplitude–frequency curves similar to those shown in Fig. 2 but with a nonlinear elastic force; $\gamma_3 = 0.2$. The numbering of the curves also corresponds to that in Fig. 2.
Values of parameters $p\eta$: $\pi/2$ (*a*), $\pi$ (*b*), $3\pi/2$ (*c*)

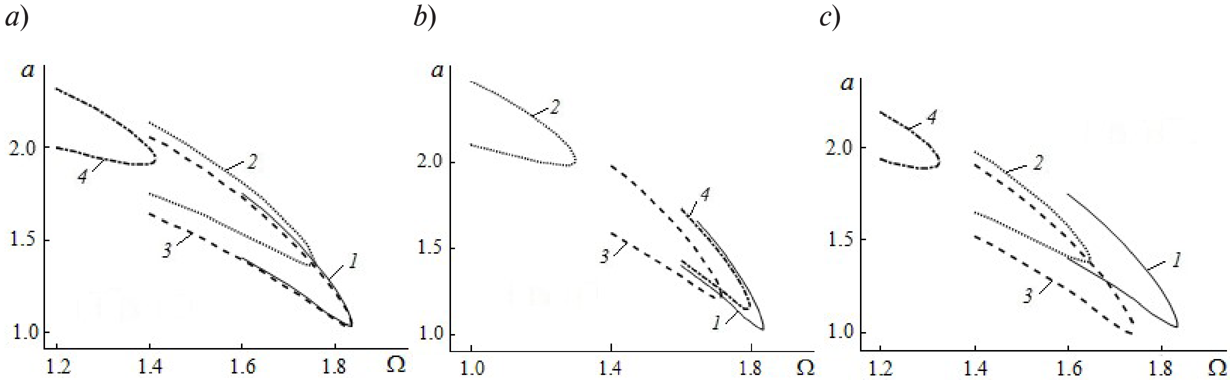


Fig. 4. Amplitude–frequency curves similar to those shown in Fig. 3, but $\gamma_3 = -0.2$. The numbering of the curves corresponds to that in Figs. 2 and 3. Values of parameters $p\eta$: $\pi/2$ (*a*), $\pi$ (*b*), $3\pi/2$ (*c*)

## Conclusion

The paper considers the dynamics of a rod with an energy source of limited power under nonlinear parametric action and a combination of delays in elasticity and damping. To obtain information about the effect of delays on the dynamics of the system, calculations were performed for stationary oscillations. The obtained results clearly illustrate the combined effect of various delays on the amplitude–frequency curves. Analysis of the obtained results allows to conclude that the delays have a significant impact on the picture of interactions:

they change the amplitude curve in the amplitude–frequency plane, shifting it right-left and up-down;

they affect the stability of oscillations.

The results of the analysis of the interaction of oscillatory systems with energy sources and the phenomena that arise in this case are described in detail in monographs [25, 26] and many other works on this area of oscillation theory. For this reason, we will not dwell on them and only note that similar effects are observed in the presence of delays.

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