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A COMPARISON OF APPROACHES TO SPECIFYING THE MODAL MATRICES IN THE MODAL CONTROL OF ELASTIC SYSTEMS WITH AND WITHOUT OBSERVERS

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Abstract. The implementation of modal control of distributed elastic objects involves the use of modal matrices: a mode analyzer and a mode synthesizer specifying the linear transformation of vectors of measured and control signals in order to separate the eigenmodes of the object in the control system. The standard method for calculating the modal matrices is the inversion of the influence matrices. The article proposes an alternative method: transposing the influence matrices with normalization of the action on different modes. As an example, the problem of suppression of forced vibrations of a thin metal beam using piezoelectric sensors and actuators has been solved numerically, and different combinations of the above methods and different variants of normalization have been tested. Two types of control systems were considered, the former being based on modal and frequency filters and the latter being based on modal observers. The best control result was shown to be achieved with the combined use of the above methods for both types of control systems considered.

Keywords: modal control, modal matrices, mode analyzer, mode synthesizer, observer

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СРАВНЕНИЕ ПОДХОДОВ К ЗАДАНИЮ МОДАЛЬНЫХ МАТРИЦ ПРИ МОДАЛЬНОМ УПРАВЛЕНИИ УПРУГИМИ СИСТЕМАМИ С НАБЛЮДАТЕЛЯМИ И БЕЗ НИХ

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Аннотация. Реализация модального управления распределенными упругими объектами предполагает использование модальных матриц – анализатора и синтезатора форм, задающих линейные преобразования векторов измеренных и управляющих сигналов с целью разделения собственных форм объекта в системе управления. Стандартный способ задания модальных матриц заключается в обращении матриц влияния. В статье предлагается альтернативный способ: транспонирование данных матриц с нормированием воздействия на разные формы. На примере численного решения задачи гашения вынужденных колебаний тонкой металлической балки с помощью пьезоэлектрических сенсоров и актуаторов тестируются разные комбинации названных методов и разные варианты нормирования. Рассмотрено управление как с наблюдателями, так и без них – на основе модальных и частотных фильтров. Показано, что наилучший результат управления достигается при комбинированном использовании рассмотренных методов в системах как с наблюдателями, так и без них.

Ключевые слова: модальное управление, модальные матрицы, анализатор форм, синтезатор форм, наблюдатель

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Introduction

In recent decades, modal control has become a widespread approach to active control of distributed systems, including elastic ones [1–3]. Its basic principle is separate control of different vibration modes of an object, assumed to be independent from each other. The efficiency of this method depends on how accurately it is possible to separate different modes of the object in the control system (CS). In control with feedback, this problem involves both accurate measurement of activation of individual modes that are controlled, and specific control actions applied to these modes.

Arrays of discrete sensors and actuators are commonly used in modal control of elastic objects. Each actuator commonly affects several modes at once, and, similarly, each sensor also reacts to several vibrational modes of the object. In this case, information about different modes in the control system is separated using modal matrices (or modal filters), setting the linear transformations of vectors of measured and control signals.

The modal approach to control can be implemented both based on modal and frequency filters [4–7], and based on observers [8–11]. In the second case, the control system turns out to be more complex, since it uses a known object model to determine the state vector of an elastic object, allowing to determine the required values more accurately.

The efficiency of these two modal approaches to damping forced vibrations in a thin metal beam was compared in our earlier paper [12]. It was established that control with observers is more efficient than control based on modal and frequency filters. Both approaches considered rely on modal matrices to separate the vibrational modes of the object in the control system.

The standard technique for calculating modal matrices consists in calculating the inverse influence matrices [5, 13] (or pseudo-inverse in the more general case [14, 15]). These matrices show the proportions in which each sensor and actuator measures or excites various modes of the object.

Thus, ideally, inversion of the influence matrices allows to obtain a system where each control loop works only with its specific eigenmode and the loops do not interfere with each other's operation. This method for calculating modal matrices was also invariably used in our earlier studies [12, 13, 16–18].

However, the given method cannot be considered a universal solution to the problem of modal separations.

Firstly, higher forms are always present, inevitably excited by the control system, since the number of modes of a distributed elastic object is infinite, while the number of controlled modes is finite. The phenomenon where energy flows to higher modes is called the spillover effect. It limits the efficiency of modal control and can lead to instability of a closed system.

Secondly, the number of sensors and actuators in the control system is also limited. If control is intended to be exercised over the number of modes exceeding the number of sensors and actuators, it is usually impossible to completely separate these modes in the control system.

In any case, ideal selection of the necessary modes during control is possible only in the rather rare situation when distributed modal sensors and actuators are used [6] (so there is no need for modal matrices); in the more common case when discrete control arrays are used, there is no universal solution to this problem, so alternative approaches should be found.

One such alternative technique for calculating modal matrices was proposed by us in [12]. It consists in transposing these matrices instead of inverting the weight coefficient matrices. They are also multiplied by additional diagonal matrices that normalize the degree of excitation and measurements of individual modes.

Instead of separating different modes to be controlled, the algorithm is primarily intended for achieving the most efficient actuation on each mode, approximating it in accordance with the influence coefficients for this particular mode.

Firstly, the proposed method is computationally simpler than the standard one, since it does not require inversion of matrices but only their transposition and multiplication by a diagonal matrix. Secondly, if the number of controlled modes changes, modal matrices do not need to be completely recalculated: it is sufficient to either add necessary columns to them or remove rows from them.

The goal of this study is to analyze the efficiency of various methods for calculating modal matrices in modal control of elastic systems.

A problem similar to the ones we discussed earlier in [12, 13, 16–18] is solved numerically for this purpose: damping of forced bending vibrations in a thin metal beam using piezoelectric sensors and actuators. Optimal control laws are synthesized for each calculation method, and the results of vibration damping for all obtained matrices are compared with each other.

Theoretical foundations of the considered control methods

Modal control is widely used to control the vibrations of elastic systems in various spheres of technology. In this paper, two of the most common approaches to modal control are investigated: a simpler one based on modal and frequency filters, and a more complex one based on observers.

This section provides a brief theoretical description of the methods under consideration (they are described in more detail in [12]), additionally substantiating various techniques for calculating modal matrices: both standard and alternative (proposed by the author of this paper).

Method of modal and frequency filters. Consider the problem of damping forced bending vibrations in a Bernoulli–Euler beam using piezoelectric sensors and actuators. Let us write the vibration equation for an elastic object in matrix form as an eigenmode expansion, assuming that vibrations in different modes occur independently:

$$\ddot{q} + 2\xi\Omega\dot{q} + \Omega^2 q = Q^c + Q^d, \tag{1}$$

where $q_{n\times1}(t)$ is the vector of generalized coordinates, its length n corresponds to the number of the object's modes taken into account in the model; $\Omega_{n\times n}$ is the diagonal matrix of natural frequencies of the vibrating beam; ξ is the scalar damping coefficient (for simplicity, we assume it to be the same for all modes); $Q^c_{n\times1}(t)$, $Q^d_{n\times1}(t)$ are the vectors of generalized forces corresponding to control and external perturbation, respectively.

Let the number of sensors and actuators be the same and equal to m ($m \le n$). Their operation is described by the following equations:

$$y_{m\times1} = \Theta^s_{m\times n} q_{n\times1},\; Q^c_{n\times1} = \Theta^a_{n\times m} u_{m\times1}, \tag{2}$$

where $y_{m\times1}(t)$ is the vector of sensor signals; $u_{m\times1}(t)$ is the vector of control signals applied to the actuators; $\Theta^s_{m\times n}$, $\Theta^a_{n\times m}$ are the influence matrices for sensors and for actuators, respectively.

If distributed modal sensors and actuators are used [6], the modes of the object are already separated in the control system: each sensor reacts, and each actuator affects only one specific mode. However, such sensors and actuators are used in exceptional cases; they are often inconvenient and too expensive, especially if several modes of the object are to be controlled simultaneously. For this reason, below we consider the case of discrete sensors and actuators.

We assume that control is carried out for k lower modes ($k \le n$), therefore, the control system includes k loops. Modal matrices T and F (mode analyzer and synthesizer) carry out linear transformations of measured and control signals in the CS. These transformations ensure that each control loop corresponds to a specific mode of the object:

$$\hat{q}_{k\times 1} = T_{k\times m} y_{m\times 1},\ u_{m\times 1} = F_{m\times k} \hat{Q}_{k\times 1}, \tag{3}$$

where $\hat{q}_{k\times 1}(t)$ is the estimate vector of k lower generalized coordinates, $\hat{Q}_{k\times 1}(t)$ is the vector of required control actions on k lower eigenmodes.

The required action on the mode in each loop of the modal system depends on the estimate of the respective generalized coordinate:

$$\hat{Q}_i = -R_i(s)\hat{q}_i, \tag{4}$$

where $R_i(s)$ is the control law in the ith circuit, written as a function of the complex variable *s*.

The control laws in the loops are also called frequency filters, set in such a way that the control system exerts the action on the object required with respect to amplitude and phase, precisely near the resonant frequency of the object corresponding to this loop.

Evidently, in the simplest case, when $k = m = n$, modal matrices should be calculated as follows:

$$T = \left(\Theta^s\right)^{-1},\ F = \left(\Theta^a\right)^{-1}. \tag{5}$$

In this case, system of equations (1) is expanded into n independent equations for each of its eigenmodes:

$$\ddot{q}_i + 2\xi\Omega_i\dot{q}_i + \Omega_i^2 q_i = -R_i(s)q_i + Q_i^d, \tag{6}$$

and efficient separate control of each eigenmode of the object can be carried out by selecting the control laws $R_i(s)$.

However, the number of modes n to be taken into account in control of distributed systems generally exceeds the number of sensors and actuators m as well as the number of modes k controlled; the numbers m and k are also not necessarily the same. In this case, the influence matrices can be represented as follows:

$$\Theta^s_{m\times n} = \begin{bmatrix} \bar{\Theta}^s_{m\times k} & \tilde{\Theta}^s_{m\times(n-k)} \end{bmatrix},\ \Theta^a_{n\times m} = \begin{bmatrix} \bar{\Theta}^a_{k\times m} \\ \tilde{\Theta}^a_{(n-k)\times m} \end{bmatrix}, \tag{7}$$

while the modal filters are defined as pseudo-inverse to the corresponding components of these matrices:

$$T_{k\times m} = \left(\bar{\Theta}^s_{m\times k}\right)^+,\ F_{m\times k} = \left(\bar{\Theta}^a_{k\times m}\right)^+. \tag{8}$$

The above method for calculating modal matrices can be considered the standard approach [14, 15, 19]. Below we discuss an alternative we proposed in [12].

Observer method. To describe this method, we represent system (1), (2) in the state space:

$$\dot{q}^n = Aq^n + Bu + Dd, \tag{9}$$

$$y = Cq^n, \tag{10}$$

where d is the vector of external influences; y, u are the vectors of the measured signals and control actions; q^n is the vector of the state of the system related as follows to the vector of generalized coordinates from Eq. (1):

$$q^n = \begin{pmatrix} q_1 & \dots & q_n & \dot{q}_1 & \dots & \dot{q}_n \end{pmatrix}^T; \tag{11}$$

matrices A, B and C can be expressed in terms of matrices describing the dynamics of the object and the operation of sensors and actuators:

$$A = \begin{bmatrix} 0_{n\times n} & I_{n\times n} \\ -\Omega^2_{n\times n} & -2\xi\Omega_{n\times n} \end{bmatrix}, B = \begin{bmatrix} 0_{n\times m} \\ \Theta^a_{n\times m} \end{bmatrix}, C = \begin{bmatrix} \Theta^s_{m\times n} & 0_{m\times n} \end{bmatrix}. \quad (12)$$

Here $0_{n\times n}$, $0_{n\times m}$ and $0_{m\times n}$ are matrices consisting of zeros; $I_{n\times n}$ is the unit matrix.

We assume that observation and control are carried out for k lower modes of the object ($k \le n$). The observer's task is to estimate the state vector q^k corresponding to these modes:

$$q^k = \begin{pmatrix} q_1 & \dots & q_k & \dot{q}_1 & \dots & \dot{q}_k \end{pmatrix}^T. \quad (13)$$

The observer generates an estimate $\hat{q}_{2k\times 1}$ of this vector using known matrices $A^{(1)}_{2k\times 2k}$, $B^{(1)}_{2k\times m}$ and $C^{(1)}_{m\times 2k}$ describing the dynamics of k lower modes of the object (they can be obtained from matrices A, B and C by removing unnecessary columns and rows):

$$\dot{\hat{q}} = A^{(1)}\hat{q} + B^{(1)}u + L\left(y - C^{(1)}\hat{q}\right), \quad (14)$$

where $L_{2k\times m}$ is the observation matrix to be calculated.

The control action depends on the estimates of the state vector:

$$u = -R\hat{q}, \quad (15)$$

where $R_{m\times 2k}$ is the control matrix that also is to be calculated.

The principle of separate control of the object's different modes dictates the following structure for the observation matrix L and control matrix R:

$$L_{2k\times k} = \begin{bmatrix} K^L_{k\times k} \\ K^{Ld}_{k\times k} \end{bmatrix} T_{k\times m}, R_{m\times 2k} = F_{m\times k}\begin{bmatrix} K^R_{k\times k} & K^{Rd}_{k\times k} \end{bmatrix}, \quad (16)$$

where K^L, K^{Ld}, K^R, K^{Rd} are diagonal matrices of size $k \times k$; T, F are modal matrices that are already known (mode analyzer and synthesizer).

Modal matrices can be calculated by the same technique as in the case of control without observers.

Alternative technique for calculating modal matrices. The idea of the method presented below was first proposed by the author of this study earlier in [12], however, theoretical substantiation as well as the first results and study of the efficiency of the proposed method are presented for the first time in this paper.

Thus, the central idea of the proposed method is to calculate modal matrices not by inversion, but rather by transposition of influence matrices. An additional mathematical operation is also performed, consisting of multiplying the resulting matrices by the diagonal matrices giving the degree of excitation and response of the control system to the object's individual modes. The proposed method is described by the following equations:

$$T_{k\times m} = M^s_{k\times k}\left(\bar{\Theta}^s_{m\times k}\right)^T, F_{m\times k} = \left(\bar{\Theta}^a_{k\times m}\right)^T M^a_{k\times k}, \quad (17)$$

where $M^s_{k\times k}$, $M^a_{k\times k}$ are the diagonal matrices to be determined.

Next, we consider different approaches to calculating these matrices.

The proposed method can be substantiated theoretically by expanding on the theory of modal filters, described in one of the fundamental studies on this subject [19]. Let us apply this theory to the problem of controlling the bending vibrations in a Bernoulli—Euler beam using piezoelectric sensors and actuators.

Consider a beam of length l located along the x axis, making bending vibrations in the XZ plane. Let us represent the transverse displacement of the points of the beam $w(x,t)$ as an expansion in terms of eigenmodes:

$$w(x,t)=\sum_{i=1}^{n}X_i(x)q_i(t), \tag{18}$$

where all the notations introduced in the previous sections are preserved, and $X_i(x)$ are the eigenmodes of the beam's bending vibrations.

We assume that the eigenmodes are normalized as follows:

$$\int_0^l \rho_l(x)X_i(x)X_j(x)dx=\delta_{ij}, \tag{19}$$

where $\rho_l(x)$ is the linear density of the beam material, δ_{ij} is the Kronecker symbol.

The normalization condition for the second derivative modes also holds true:

$$\int_0^l EI(x)X_i''(x)X_j''(x)dx=\Omega_i^2\delta_{ij}, \tag{20}$$

where $EI(x)$ is the bending stiffness of the beam's cross sections.

The operation of sensors and actuators is described by Eq. (2). We consider them as rectangular piezoelectric plates glued on both sides of the beam in sensor-actuator pairs.

In this case, the following relations hold true for the influence coefficients:

$$\Theta_{ij}^s=k^s\Theta_{ij},\ \Theta_{ji}^a=k^a\Theta_{ij}, \tag{21}$$

$$\Theta_{ij}=X_j'\left(x_i^{(2)}\right)-X_j'\left(x_i^{(1)}\right)=\int_{x_i^{(1)}}^{x_i^{(2)}}X_j''(x)dx\approx X_j''(x_i)l_p, \tag{22}$$

where k^s, k^a are the coefficients for sensors and actuators, respectively, depending on their geometric parameters and material properties; $x_i^{(1)}$, $x_i^{(2)}$, x_i are the coordinates the left and right ends as well as the center of the ith sensor–actuator pair, respectively; is the length of each piezoelectric element.

These quantities are related as follows:

$$x_i=\frac{x_i^{(1)}+x_i^{(2)}}{2},\ l_p=x_i^{(2)}-x_i^{(1)}. \tag{23}$$

Eq. (22) uses the assumption that the length of the sensors and actuators is small, allowing to approximately replace the integral of the second derivative of the beam's eigenmodes along the length of the piezoelectric element with the value of this function in the midsection. The influence coefficients essentially turn out to be proportional to the curvature of individual eigenmodes in the given cross sections. The reason for this is that longitudinal deformation of sensors and actuators is associated precisely with bending deformation (i.e., curvature) of the corresponding sections of the beam.

It is proposed in [19] to calculate the approximation of the transverse displacement function $\hat{w}(x,t)$ of beam points in terms of estimates of this function in separate cross sections corresponding to the location of sensors, $\hat{w}(x_j,t)$:

$$\hat{w}(x,t)=\sum_{j=1}^{m}G(x,x_j)\hat{w}(x_j,t), \tag{24}$$

where $G(x,x_j)$ are the interpolation functions given for each sensor.

The alternative technique for calculated the matrices, presented in this paper, assumes a slightly modified definition for the function $\hat{w}(x,t)$:

$$\hat{w}(x,t)=\sum_{j=1}^{m}\overline{G}(x,x_j)y_j(t). \tag{25}$$

A modification introduced here is that the actual signals $\hat{w}\left(x_j,t\right)$ of the sensors are used instead of the displacement estimate $y_j\left(t\right)$ in the points where the sensors are mounted. This substitution is made because the sensors in the given problem measure not the transverse displacement of the beam's cross sections, as is the case in [19], but the curvature of the beam in these cross sections, therefore it is impossible to obtain an estimate of the displacement $\hat{w}\left(x_j,t\right)$ from their signals. Therefore, the function $\bar{G}\left(x,x_j\right)$ takes on a slightly different meaning than the function $G\left(x,x_j\right)$ in the original formula.

The formula for determining the interpolation functions $\bar{G}\left(x,x_j\right)$ plays the central role in developing the proposed method.

We propose the following definition:

$$\bar{G}\left(x,x_j\right)=\sum\nolimits_{i=1}^{k}\bar{M}_i^s X_i\left(x\right)X_i''\left(x_j\right). \tag{26}$$

Thus, the interpolation function for each of the sensors is a combination of the beam's eigenmodes, where the weight of each eigenmode is proportional to the curvature of this eigenmode in the beam cross section corresponding to this sensor (i.e., in fact corresponding to the influence coefficient Θ_{ji}). In addition, the contribution of various modes is regulated by coefficients $\bar{M}_i^s$. Next, let us consider several approaches to determining them.

The following expression for the estimate of the *i*th generalized coordinate can be obtained from the condition of orthogonality and normalization of eigenmodes (19):

$$\hat{q}_i\left(t\right)=\int_0^l \rho_l\left(x\right)X_i\left(x\right)\hat{w}\left(x,t\right)dx. \tag{27}$$

Substituting expressions (25) and (26) here, and taking into account condition (19) and the definition of the mode analyzer (3), we obtain the following expression for the components of this matrix:

$$T_{ij}=\int_0^l \rho_l\left(x\right)X_i\left(x\right)\bar{G}\left(x,x_j\right)dx=\bar{M}_i^s X_i''\left(x_j\right)=\frac{\bar{M}_i^s}{l_p}\Theta_{ji}=\frac{\bar{M}_i^s}{l_p k^s}\Theta_{ji}^s. \tag{28}$$

Thus, we have obtained an expression for the matrix *T*, which coincides with Eq. (17). In this case, the coefficients are related as follows:

$$\bar{M}_i^s=l_p k^s M_i^s. \tag{29}$$

Next, let us discuss how these normalizing coefficients are to be determined. To do this, recall the standard definition of the matrix *T*, that is, Eq. (5). It assumes that the diagonal of the matrix, which is the product of the matrices *T* and Θ^s, is composed of units, i.e., the following equality holds true:

$$\sum\nolimits_{j=1}^{m}T_{ij}\Theta_{ji}^s=\frac{k^s}{l_p}\bar{M}_i^s\sum\nolimits_{j=1}^{m}\left(\Theta_{ji}\right)^2=1. \tag{30}$$

This leads to the first method for normalization, consisting of fitting:

$$\bar{M}_i^s=\frac{l_p}{k^s}\cdot\frac{1}{\sum_{j=1}^{m}\left(\Theta_{ji}\right)^2}=\frac{l_p}{k^s}M_i,\ M_i^s=\frac{\bar{M}_i^s}{l_p k^s}=\frac{1}{\left(k^s\right)^2}M_i, \tag{31}$$

where the components of the diagonal normalizing matrix are introduced:

$$M_i=\frac{1}{\sum_{j=1}^{m}\left(\Theta_{ji}\right)^2}. \tag{32}$$

The matrix synthesizing the modes is found by calculating the normalizing matrix M^a from Eq. (17) similarly to expression (31):

$$M_i^a = \frac{1}{\left(k^a\right)^2} M_i. \quad (33)$$

The second approach to calculating the normalizing matrices is asymptotic. It is formulated assuming that there is a sufficiently large number of sensors and actuators, and the sum of the squared second derivatives of the eigenmodes can be replaced by the average value of this function along the length of the beam multiplied by the number of piezoelectric elements:

$$\sum_{j=1}^{m} \left(\Theta_{ji}\right)^2 = l_p^2 \sum_{j=1}^{m} \left(X_i''(x_j)\right)^2 \approx \frac{l_p^2 m}{l} \int_0^l \left(X_i''(x)\right)^2 dx = \frac{l_p^2 m \Omega_i^2}{lEI}. \quad (34)$$

In this case, the components of the diagonal normalizing matrix take the following form:

$$M_i = \frac{EIl}{\Omega_i^2 l_p^2 m}. \quad (35)$$

Finally, another normalization method is trivial (in fact, the absence of normalization), where the matrix M is assumed to be singular:

$$M = I_{k \times k}. \quad (36)$$

Regardless of the method for determining the normalizing matrix, the proposed method for calculating modal matrices is much simpler than the standard method (8) from the standpoint of calculations. It also has another advantage: if the number of controlled modes is changed (while maintaining the sensor and actuator system) there is no need to completely recalculate the modal matrices, as with the standard method; it is sufficient to either add the corresponding columns to them or remove rows from them.

This raises the natural question whether the standard and alternative approaches to calculating modal matrices are fundamentally different; if the answer is yes, the conundrum is how this is possible since both approaches are aimed at solving the same problem.

This apparent contradiction is explained simply: if the number of sensors and actuators, evenly distributed over the control object, is increased, the results yielded by these approaches converge, since in this case the columns of the matrix Θ become orthogonal to each other, and the results of inverting this matrix and transposing it (provided that the first normalization method (32) is used) become identical.

In view of this remark, the greatest difference between the considered approaches to calculating modal matrices is manifested in the case when the number of sensors and actuators used turns out to be small. This specific case is considered below: the number of pairs of piezoelectric elements in the problem is taken equal to two.

In addition to the formulation of the proposed approach, the novelty of this study is that we consider the case when the number of modes controlled exceeds the number of sensor-actuator pairs ($k > m$), while the opposite situation is traditionally considered in the literature on modal control ($k \leq m$) [14, 15]. This means that the modes controlled in the systems discussed below cannot be strictly separated from each other, which further complicates the problem of determining modal matrices and increases the importance of studying the alternative approaches to solving this problem.

Problem statement

The problem solved in this study involves damping of forced bending vibrations in a thin aluminum beam elastically restrained in the midsection by piezoelectric sensors and actuators glued to the beam in certain regions. This problem has already been considered in some of our earlier studies: first in an experimental study [16], then in numerical ones [12, 17, 18], where the goal was to simulate the processes occurring in the experiment as accurately as possible.

The experimental setup is shown schematically in Fig. 1. Beam *1* with a cross section of 3 × 35 mm and a length of 70 cm is arranged vertically and fixed at one point at a distance of 10 cm from the lower end. Piezoelectric stack actuator *2* is included in the fastener connecting the

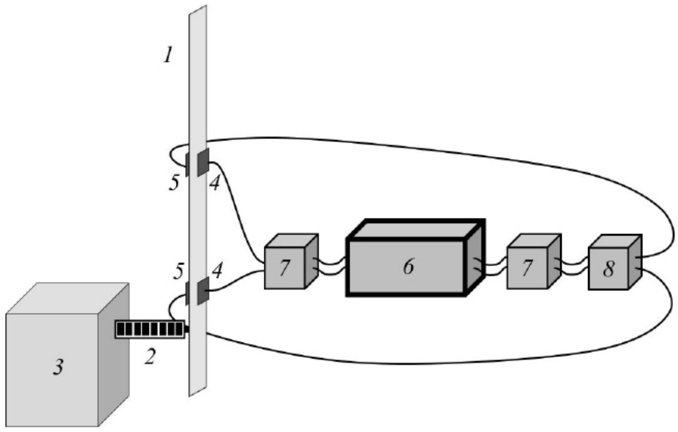

Fig. 1. Schematic of experimental setup:
aluminum beam 1; piezoelectric stack actuator 2;
fixed base 3; sensors 4; actuators 5;
digital controller 6; low-pass filters 7; amplifier 8

beam to fixed base *3*. Longitudinal vibrations of the actuator, occurring when an AC voltage is applied to it with a certain frequency, induce vibrations in the point where the beam is connected. Such vibration acts as external excitation, whose consequences the control system should mitigate.

The control system consists of two sensor-actuator pairs (sensors *4*, actuators *5*), which are thin rectangular piezoelectric plates measuring 50 × 30 mm, covered with electrodes on both sides.

In addition to digital controller *6*, converting the measured signals into control signals, the control circuit includes additional elements: low-pass filters *7* and amplifier *8*. Filters smooth out the high-frequency components of the signal arising from discretization in the controller, and generally increase the stability of the closed system; the amplifier increases the amplitude of the control signal by 25 times before it is fed to the actuators.

The frequency characteristics of the filters and the amplifier are also taken into account in numerical simulation of a closed system [12], which distinguishes this study from most numerical studies on this subject.

Initially, the initial goal of control in the experiment and subsequent numerical studies, where control systems were synthesized without observers, was to dampen the forced bending vibrations of the beam only at the first and second resonances. The arrangement of sensors and actuators on the beam was chosen in accordance with this goal [16]. However, our most recent studies, starting with [12], consider, among other things, more efficient modal systems with observers allowing to increase the number of beam eigenmodes to be controlled, while preserving the number and arrangement of sensors and actuators. Therefore, the goal of control for such systems was formulated as damping forced beam vibrations at three lower resonances.

The efficiency of control in the experiment was monitored with a laser vibrometer measuring the vibration amplitude of a point at the upper end of the beam (since it is this point that experiences the largest displacements during forced vibrations of the beam).

A finite element model of the control object (beams with piezoelectric elements and fastener) was constructed at the first stage of numerical simulation of the closed system; analysis of the model allows to obtain the frequency characteristics of the object [17]. Next, the frequency characteristics of the closed system were calculated for each type of control system tested based on the frequency characteristics of both the object and the CS itself [17]. To determine the efficiency of control, we analyzed the frequency response of displacement of a point at the upper end of the beam.

Synthesis of control systems

This study was aimed at synthesizing optimal control systems within the framework of the approaches considered, i.e., such system that achieve the goal of control posed the most efficiently.

For this purpose, an optimization procedure is used to construct the control systems [17, 18], allowing to vary the parameters of the control system and select their optimal combinations, satisfying the condition for stability of the closed system.

The optimization criterion is either the height of a specific resonant peak on the frequency response of the beam (corresponding to displacement of the point at the upper end), or the sum of the vibration amplitudes (in decibels) at the corresponding resonant peaks in the case when the system must dampen vibrations at several resonances.

The stability of the closed system was analyzed using the Nyquist criterion, modified for the case of several control loops [17].

Modal matrices, a mode analyzer and a mode synthesizer (T and F) are calculated at the first stage of CS synthesis, both with and without observers. They are calculated by different techniques, discussed above. Then, control laws are synthesized for each combination of modal matrices using the optimization procedure: for systems without observers, these are frequency filters $R_1(s)$ and $R_2(s)$ (see Eq. (4)), and for systems with observers, these are diagonal matrices K^L, K^{Ld}, K^R and K^{Rd}, included in the definition of observation and control matrices L and R (see expressions (16)). Various goals of control are set: both the damping of beam vibrations at each resonance separately, and simultaneous damping at several resonances.

Let us focus more closely on different techniques for calculating the modal matrices used in this study. We assume that equalities (21) hold true for the influence matrices. In this case, two approaches to calculating the matrices are globally possible.

The first method is inversion (or pseudo-inversion) (8):

$$T = \left(\Theta^s\right)^+ = k^s\Theta^+,\ F = \left(\Theta^a\right)^+ = k^a\left(\Theta^T\right)^+. \quad (37)$$

The second method is transposition with multiplication by the matrix M (see expressions (31, 33)):

$$T = \frac{1}{\left(k^s\right)^2} M\left(\Theta^s\right)^T = \frac{1}{k^s} M\Theta^T,\ F = \frac{1}{\left(k^a\right)^2}\left(\Theta^a\right)^T M = \frac{1}{k^a}\Theta M. \quad (38)$$

The following influence matrices were obtained from the object model for cases of control over two or three modes:

$$\Theta_{2\times2} = \begin{bmatrix} 3.659 & -17.07 \\ 1.187 & 17.76 \end{bmatrix},\ \Theta_{2\times3} = \begin{bmatrix} 3.659 & -17.07 & 21.48 \\ 1.187 & 17.76 & 10.09 \end{bmatrix}. \quad (39)$$

The normalizing matrix M was determined by one of the three techniques (see Eqs. (32), (35), (36)) (the last row and column are removed for control over two modes):

$$\begin{cases} M^{triv} = I_{3\times3},\ M^{asimp} = diag\{0.574 \quad 0.0159 \quad 0.00223\}, \\ M^{fit} = diag\{0.0676 \quad 0.00165 \quad 0.00178\}. \end{cases} \quad (40)$$

In this case, the product of multiplying the modal matrices by the corresponding influence matrices is of greater interest for analysis than the matrices themselves; let us denote it as $\tilde{\Theta}$. The matrix $\tilde{\Theta}$ should have as much similarity as possible to the unit matrix for the best separation of modes.

This requirement is satisfied for the case of control over two modes ($k = 2$) and inversion of the influence matrices:

$$\tilde{\Theta}^{inv}_{2\times2} = T\Theta^s = \left(\Theta^a F\right)^T = I_{2\times2}. \quad (41)$$

Below we present the matrices corresponding to pseudo-inversion and different normalization methods for transposition with $k = 3$: trivial, asymptotic and fitting (matrices for $k = 2$ can be obtained by removing the last row and column from these matrices):

$$\begin{cases} \tilde{\Theta}^{inv} = \begin{bmatrix} 0.0236 & -0.0201 & 0.150 \\ -0.0201 & 1.00 & 0.00310 \\ 0.150 & 0.00310 & 0.977 \end{bmatrix}, \; \tilde{\Theta}^{triv} = \begin{bmatrix} 14.8 & -41.4 & 90.6 \\ -41.4 & -607 & -187 \\ 90.6 & -187 & 563 \end{bmatrix}, \\ \tilde{\Theta}^{asimp} = \begin{bmatrix} 8.49 & -23.8 & 52.0 \\ -0.658 & 9.65 & -2.98 \\ 0.202 & -0.417 & 1.25 \end{bmatrix}, \; \tilde{\Theta}^{fit} = \begin{bmatrix} 1.00 & -2.80 & 6.12 \\ -0.0682 & 1.00 & -0.309 \\ 0.161 & -0.333 & 1.00 \end{bmatrix}. \end{cases} \quad (42)$$

As evident from the formulas obtained, different techniques for calculating the modal matrices produce significantly different results for separation of the beam's eigenmodes in the control system. Next, we can compare the efficiency of the considered approaches.

Comparison of results obtained from different control systems

This section presents the results from the synthesized control systems. Examples of control laws for the CS based on the method of modal and frequency filters are given in [18], and examples for the CS with observers are given [12].

Results of damping of forced beam vibrations at the first (I) and second (II) resonances for systems without observers are given in Table 1. If the modal matrices are calculated by the transposition method in such CS, it does not matter which normalization method is used, since different methods produce the same result with the appropriate choice of gain coefficients in the control laws in each of the loops. The CS synthesized for each technique for calculating modal matrices (inversion or transposition) were efficient only at the first or second resonances separately or at both resonances together.

As evident from the data presented, the most efficient damping of vibrations at both resonances is achieved when one of the modal matrices is determined by inversion, and the other by transposition (cases 3 and 4), the latter case is slightly more efficient.

Table 1

Decrease in resonant amplitudes of beam vibrations for various control systems (without observers)

Calculation technique			Damping	Amplitude decrease, dB, at resonance	
Case	T	F		I (Δy_1)	II (Δy_2)
1	Tr	Inv	separately	32.04	30.84
			together	32.00	31.05
2	Tr	Tr	separately	32.31	31.41
			together	32.32	29.9
3	Inv	Tr	separately	32.72	31.48
			together	32.72	31.48
4	Tr	Inv	separately	32.77	31.45
			together	32.77	31.50

Notations: T, F are the mode analyzer and synthesizer, respectively; Tr, Inv are transposition and inversion of influence matrices, respectively.

Table 2 gives the results obtained from different CS with observers at the first three resonances. Two cases are considered for each combination of modal matrices: control over only two (k = 2) or over all three modes (k = 3). In each of these cases, various normalization methods were considered: all three methods for k = 3, only trivial and asymptotic for k = 2, since in this case there is practically no difference between the asymptotic and fitting cases. The best results of vibration damping at each of the resonances for each combination of modal matrices are highlighted in bold in Table 2.

Table 2

Decrease in resonant amplitudes of beam vibrations for various control systems (with observers)

Calculation technique			k	Normalization	Amplitude decrease, dB, at resonance		
Case	T	F			I (Δy_1)	II (Δy_2)	III (Δy_3)
1	Inv	Inv	2	–	34.75	35.66	-
			3	–	**36.50**	**36.52**	**22.85**
2	Tr	Tr	2	*triv*	36.44	36.55	–
				asymp	**36.68**	36.53	–
			3	*triv*	35.48	**37.02**	**23.91**
				asymp	36.33	36.95	23.88
				fit	36.18	36.99	22.75
3	Inv	Tr	2	*triv*	35.97	31.76	–
				asymp	36.06	31.86	–
			3	*triv*	36.18	37.09	**21.99**
				asymp	**36.78**	38.23	21.69
				fit	36.73	**38.33**	21.83
4	Tr	Inv	2	*triv*	36.93	36.98	–
				asymp	36.94	37.18	–
			3	*triv*	37.68	**37.65**	24.28
				asymp	37.86	37.62	24.35
				fit	**37.93**	37.64	**24.46**

Notations: k is the number of modes; *triv*, *asimp*, *fit* are trivial, asymptotic and fitting normalization methods used for transposition.

Fig. 2 shows the frequency response of the closed system, where the vibration amplitude of the point at the upper end of the beam, near each of the three lower resonances, acts as the observable. The efficiency of four CS corresponding to each of the combinations of modal matrices in the case of fitting normalization with simultaneous damping of vibrations at three resonances is compared.

Notably, the difference between all the considered calculation techniques is small: it is within a few decibels for control with observers, and about a decibel for control without observers. Thus, robustness is observed in this aspect of the approaches to control considered. The most efficient combinations of matrices for systems with observers are also 3 and 4, and the latter is again preferable: it allows to damp the first and third resonances, while combination 3 allows to damp the second resonance.

On average, the results for cases of asymptotic and fitting normalization are better than for trivial the trivial normalization: asymptotic normalization is better for combination 2, and fitting normalization is better for combinations 3 and 4. In general however, the normalization method is less important than the technique for calculating the matrices (inversion or transposition). It is also worth noting that, on average, the first two resonances are damped better with control over three modes ($k = 3$) than over two modes ($k = 2$).

In addition, the study confirms the conclusion drawn in [12]: control with observers remains significantly more efficient than that based on modal and frequency filters for any techniques for calculating modal matrices, besides it allows to damp vibrations at a larger number of resonances exceeding the number of sensors and actuators.

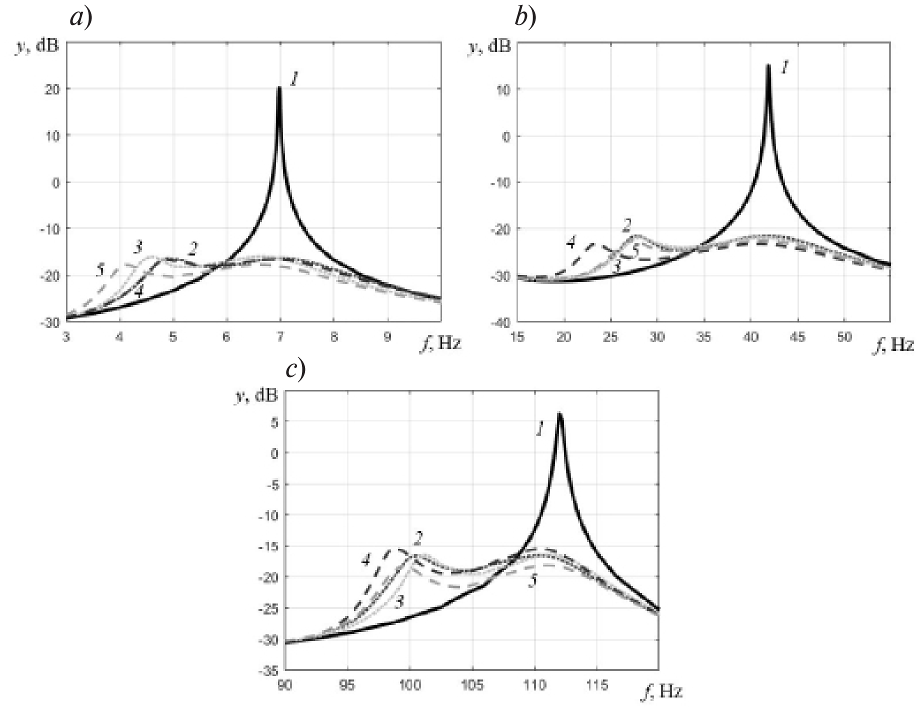

Fig. 2. Frequency response of the beam for different control systems (with observers) near resonances I (*a*), II (*b*) and III (*c*): without control (curves *1*) and for different techniques 1–4 used to calculate modal matrices (curves *2*–*5*) (see Table 2 for calculation techniques 1–4)

Conclusion

The paper considers various techniques for calculating modal matrices within the modal approach to control of distributed systems. Along with the traditional method involving inversion of influence matrices, an alternative method was considered and substantiated, consisting in transposition of these matrices with subsequent normalization. The example of the numerical solution to the problem of damping of forced bending vibrations in a thin metal beam was used to prove that the best result of control was achieved by combining the given methods, when the mode analyzer matrix was calculated by transposition, and the mode synthesizer matrix was calculated by inversion. This conclusion is valid for systems both with observers and based on modal and frequency filters. In addition, it is established that the results of control can be improved through normalization of modal matrices calculated by transposition.

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