Mathematics

Original article
UDC 515.1.
DOI: https://doi.org/10.18721/JPM.16417

FIXED POINT THEOREMS ON ORTHOGONAL METRIC SPACES VIA τ-DISTANCES

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Abstract. In this paper, we prove two fixed point theorems in the setting of orthogonal complete metric spaces via τ-distances. Our theorems generalize and improve many known results in the literature (see, for example Refs. [6, theorem 4.2] and [3, theorem 3]).

Keywords: fixed point, orthogonal generalized E-weakly contractive maps, orthogonal metric space, Hausdorff topological spaces, τ-distance

For citation: Touail Y., Jaid A., El Moutawakil D., Fixed point theorems on orthogonal metric spaces via τ-distances, St. Petersburg State Polytechnical University Journal. Physics and Mathematics. 16 (4) (2023) 215–223. DOI: https://doi.org/10.18721/JPM.16417

Научная статья
УДК 515.1.
DOI: https://doi.org/10.18721/JPM.16417

ТЕОРЕМЫ О НЕПОДВИЖНОЙ ТОЧКЕ НА ОРТОГОНАЛЬНЫХ МЕТРИЧЕСКИХ ПРОСТРАНСТВАХ, ДОКАЗАННЫЕ С ПОМОЩЬЮ ПОНЯТИЯ τ-РАССТОЯНИЯ

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Аннотация. В этой статье мы доказываем две теоремы о неподвижной точке в задании ортогональных полных метрических пространств, используя понятие τ-расстояния. Выдвинутые и доказанные теоремы позволяют обобщить и улучшить многие известные результаты, опубликованные в литературе (см., например, результаты в статьях [6, теорема 4.2] и [3, теорема 3]).

Ключевые слова: неподвижная точка, ортогональное обобщенное E-слабосжимаемое отображение, ортогональное метрическое пространство, хаусдорфово топологическое пространство, τ-расстояние

Для цитирования: Туай Ю., Джайд А., Аль-Мутавакиль Д. Теоремы о неподвижной точке на ортогональных метрических пространствах, доказанные с помощью понятия τ-расстояния // Научно-технические ведомости СПбГПУ. Физико-математические науки. 2023. Т. 16. № 4. С. 215–223. DOI: https://doi.org/10.18721/ JPM.16417

Introduction

In 2003, M. Aamri and D. El Moutawakil [1] introduced the concept of τ-distance in general topological spaces. This innovation has extended a lot of ideas about known spaces presented in the literature. Moreover, these scientists proved a version of the Banach's fixed point theorem for this general setting.

In 2017, M. E. Gordji et al. [2] defined so-called orthogonal metric spaces as a generalization of the metric spaces. The authors showed in Ref. [2] that this type of spaces is very powerful and applicable to many cases, such as the fixed point theory. Then an important extension of Banach's fixed point theorem was given.

Without using the compactness of the space, the author of Ref. [6] put forward some fixed point theorems for new classes of mappings via τ-distance in general topological spaces (some related results can be found in Refs. [3 – 5, 7]).

In this paper, motivated by Refs. [2, 6], we extend some results proven in Ref. [6]; in other words, we will restrict our studies to the orthogonal elements only, in order to prove the fixed point property for a large class of contractive mappings. Our results will be based specially on some essential notions like orthogonality, τ-distances in the general topological spaces. Some important examples will also be given to support the proven theorems and to show the usability of this new direction of research.

Preliminaries

The aim of this section is to present some concepts and known results used in the paper.

Let (X, τ) be a topological space and $p: X \times X \to [0, +\infty)$ be a function. For any $\varepsilon > 0$ and any $x \in X$, let $B_p(x, \varepsilon) = \{y \in X \,/\, p(x, y) < \varepsilon\}$.

Definition I [1, definition 2.1]. The function p is said to be τ-distance if there exists $\varepsilon > 0$ for each $x \in X$ and any neighborhood V of x, such that $B_p(x, \varepsilon) \subset V$.

Definition II. In a Hausdorff topological space X, a sequence $\{x_n\}$ is said to be a p-Cauchy sequence if it satisfies the usual metric condition with respect to p; in other words, if $\lim_{n,m\to\infty} p(x_n, x_m) = 0$.

Definition III [1, definition 3.1]. Let (X, τ) be a topological space with a τ-distance p.

1. X is S-complete if there exists x in X for every p-Cauchy sequence (x_n), such that $\lim p(x, x_n) = 0$.
2. X is considered p-Cauchy complete if there exists x in X for every p-Cauchy sequence (x_n), such that $\lim x_n = x$ with respect to τ.
3. X is said to be p-bounded if $\sup\{p(x, y) \,/\, x, y \in X\} < \infty$.

Lemma 1 [1, lemma 3.1]. *Let* (X, τ) *be a Hausdorff topological space with a τ-distance p, then*

1) $p(x, y) = 0$ *implies* $x = y$.

2) Let (x_n) *be a sequence in X such that lim* $p(x, x_n) = 0$ *and* $\lim p(y, x_n) = 0$, *then* $x = y$.

Lemma 1 was proved in Ref. [1].

Definition IV. [1, definition 2.5]). Ψ is the class of all functions ψ from [0, +∞) to [0, +∞) satisfying:

i) ψ is nondecreasing,

ii) $\lim \psi^n(t) = 0$ for all $t \in [0, +\infty)$.

Definition V. Φ is the class of all functions ϕ from [1, +∞) to [0, +∞) satisfying:

i) $\phi(t) = 0$ if and only if $t = 1$,

ii) $\inf_{t>1} \phi(t) = 0$.

Theorem 1. [1, theorem 4.1]. *Let* (X, τ) *be a Hausdorff topological space with a τ-distance p. Suppose that X is p-bounded and S-complete. Let T be a selfmapping of X such that*

$$p(Tx, Ty) \le \phi(p(x, y)),$$

for all x, y ∈ *X. Then T has a unique fixed point.*

Theorem 1 was proved in Ref. [1].

Theorem 2 [6]. *Let* (X, τ) *be a Hausdorff topological space with a* τ*-distance p. Suppose that X is p-bounded and S-complete. Let T be a p-continuous selfmapping of X such that*

$$p(Tx,Ty) \le \phi\Big(\max\big\{p(x,y),p(x,Tx),p(y,Ty)\big\}\Big), \quad (1)$$

f*or all* $x, y \in X$. *Then T has a unique fixed point.*

Theorem 2 was proved in Ref. [6].

Theorem 3 [6]. *Let* $T: X \to X$ *be a generalized E-weakly contractive mapping of a bounded complete metric space* (X, d). *Then T has a unique fixed point.*

Theorem 3 was proved in Ref. [6].

Theorem 4 [6]. *Let* $T: X \to X$ *be a mapping of a bounded complete metric space* (X, d) *such that*

$$\inf_{x\neq y\in X}\Big\{\max\big\{d(x,y),d(x,Tx),d(y,Ty)\big\}-d(Tx,Ty)\Big\} > 0. \quad (2)$$

Then T has a unique fixed point.

Theorem 4 was proved in Ref. [6].

Now we recall the definition of an orthogonal set and some related basic notions.

Definition VI [2]. *Let* $X \neq \phi$ and let $\perp \subset X \times X$ be a binary relation. If $\perp$ satisfies the following hypothesis:

$$\exists x_0 : (\forall y, y \perp x_0) \text{or} (\forall y, x_0 \perp y), \quad (3)$$

then it called an orthogonal set (briefly *O*-set); we denote this *O*-set by $(X, \perp)$.

Note that x_0 is said to be an orthogonal element in the Definition VI.

Remark. In general, x_0 is not unique, otherwise, $(X, \perp)$ is called unique orthogonal set and the element x_0 is said to be a unique orthogonal element.

Definition VII [2]. Let$(X, \perp)$ be an *O*-set. A sequence $\{x_n\}$ is called an orthogonal sequence (briefly, *O*-sequence) if

$$(\forall n, x_n \perp x_{n+1}) \text{or} (\forall n, x_{n+1} \perp x_n).$$

Definition VIII [2]. The triplet $(X, \perp, d)$ is called an orthogonal metric space if (X, d) is a metric space and $(X, \perp)$ is the *O*-set.

Definition IX [2]. Let $(X, \perp, d)$ be an orthogonal metric space. Then, a mapping $T: X \to X$ is said to be orthogonally continuous (briefly $\perp$-continuous) in $x \in X$, if for each *O*-sequence $\{x_n\} \subset X$ such that $x_n \to x$ as $n \to \infty$, we obtain $Tx_n \to Tx$ as $n \to \infty$. *T* is said to be $\perp$-continuous on *X* if *T* is $\perp$-continuous in each $x \in X$ as well.

Definition X [2]. Let $(X, \perp, d)$ be an orthogonal metric space. Then, *X* is said to be orthogonally complete (or $\perp$-complete) if every Cauchy *O*-sequence is convergent.

Definition XI [2]. Let $(X, \perp)$ be the *O*-set. A mapping $T: X \to X$ is said to be $\perp$-preserving if $Tx \perp Ty$ whenever $x \perp y$.

Remark [2]. Every complete metric space (continuous mapping) is *O*-complete metric space ($\perp$-continuous mapping) and the converse is not true.

Theorem 5 [2]. *Let* $(X, \perp, d)$ *O-complete metric space and T a self-mapping on X which is* $\perp$*-preserving and* $\perp$*-continuous. If there exists* $k \in [0.1)$ *such that for all* $x, y \in X$

$$x \perp y \text{ implies } d(Tx, Ty) \le kd(x, y).$$

Then T has a unique fixed point.

Theorem 5 was proved in Ref. [2].

Now, we give some examples of orthogonal spaces.

Example 1 [2]. Let $X = \mathbf{Z}$. Define the binary relation $\perp$ on *X* by $m \perp n$ if there exists $k \in \mathbf{Z}$ such that $m = kn$. It is easy to see that $0 \perp n$ for all $n \in \mathbf{Z}$. Hence, $(X, \perp)$ is the *O*-set.

Example 2 [2]. Let *X* be an inner product space with the inner product. Define the binary relation $\perp$ on *X* by $x \perp y$ if $(x, y) = 0$. It is easy to see that $0 \perp x$ for all $x \in X$. Hence, $(X, \perp)$ is the *O*-set.

For more details, we refer the reader to see Ref. [2].

Main results

In this section, we start with some definitions and lemmas.

Definition XII. The triplet $(X, \perp, d)$ is called an orthogonal Hausdorff topological space with a τ-distance p if (X, τ) is a Hausdorff topological space with a τ-distance p and $(X, \perp)$ is an orthogonal set.

Definition XIII. Let (X, τ) be a topological space with a τ-distance p. Then T: $X \to X$ is said to be orthogonal p-continuous at $x \in X$ if we have for any orthogonal $\{x_n\} \subset X$ such that $\lim p(x, x_n) = 0$.

Lemma 2. *Let $(X, \perp, d)$ be an orthogonal Hausdorff topological space with a τ-distance p such that $p(x, x) = 0$ for all $x \in X$. Suppose that X is p-bounded and S-complete. Let T be a $\perp$-continuous and $\perp$-preserving self-mapping of X such that $x \perp y$ implies*

$$p(Tx,Ty) \le \phi\Big(\max\big\{p(x,y), p(x,Tx)\, p(y,Ty)\big\}\Big), \tag{4}$$

for all $x, y \in X$, where $\phi \in \Phi$. Then has a unique fixed point.

P r o o f. Since X is an orthogonal set, there exists at least $x_0 \in X$ such that

$$(\forall y, y \perp x_0) \text{or} (\forall y, x_0 \perp y). \tag{5}$$

This implies that $x_0 \perp Tx_0$ or $Tx_0 \perp x_0$ Consider the iterated sequence $\{x_n\}$ such that $x_n = T^n x_0$ for all $n \in N$. As T is a $\perp$-preserving, we obtain either $T^n x_0 \perp T^{n+1} x_0$ or $T^{n+1} x_0 \perp T^n x_0$ for all $n \in N$. Then $\{x_n\}$ is an O-sequence.

Let $n \in N$

$$p(x_{n+1}, x_{n+2}) \le \phi\Big(\max\big\{p(x_n, x_{n+1}), p(x_n, x_{n+1}), p(x_{n+1}, x_{n+2})\big\}\Big) \le$$
$$\le \phi\Big(\max\big\{p(x_n, x_{n+1}); p(x_{n+1}, x_{n+2})\big\}\Big).$$

If there exists $n \in N$ for which $p(x_{n_0}, x_{n_0+1}) < p(x_{n_0+1}, x_{n_0+2})$, then $p(x_{n_0+1}, x_{n_0+2}) < p(x_{n_0+1}, x_{n_0+2})$, this leads to contradiction.

Then $p(x_{n+1}, x_{n+2}) < p(x_n, x_{n+1})$ for all $n \in N$ which implies that

$$p(x_{n+1}, x_{n+2}) < \phi\big(p(x_n, x_{n+1})\big) \tag{6}$$

for every $n \in N$.

Now, let $n, m \in N$, we obtain from formula (5) $x_0 \perp x_n$ or $x_n \perp x_0$, using the fact that T is $\perp$-preserving, we get $x_n \perp x_{n+m}$ or $x_{n+m} \perp x_n$, which implies by inequality (6) that

$$\begin{aligned} p(x_n, x_{n+m}) &= p(\mathrm{T}x_{n-1}, \mathrm{T}x_{n+m-1}) \le \\ &\le \phi\Big(\max\big\{p(x_{n-1}, x_{n+m-1}), p(x_{n-1}, x_n), p(x_{n+m-1}, x_{n+m})\big\}\Big) \le \\ &\le \phi\Big(\max\big\{p(x_{n-1}, x_{n+m-1}), p(x_{n-1}, x_n)\big\}\Big) \le \\ &\le \phi\Big(\max\Big\{\phi\Big(\max\big\{p(x_{n-2}, x_{n+m-2}), p(x_{n-2}, x_{n-1})\big\}\Big), \phi\big(p(x_{n-2}, x_{n-1})\big)\Big\}\Big) \le \\ &\le \phi^2\Big(\max\big\{p(x_{n-2}, x_{n+m-2}), p(x_{n-2}, x_{n-1})\big\}\Big) \le \\ &\vdots \\ &\le \phi^n\Big(\max\big\{p(x_0, x_m), p(x_0, x_1)\big\}\Big) \le \\ &\le \phi^n(M), \end{aligned} \tag{7}$$

where $M = \sup\{p(x, y) / x, y \in X\}$.

Letting $n \to \infty$ in formula (7), we deduce that $\{x_n\}$ is an orthogonal p-Cauchy sequence. Since X is an orthogonal S-complete space, there exists $u \in X$ such that $\lim p(u, x_n) = 0$.

On the other side, the orthogonal p-continuity of the mapping T implies that $\lim p(Tu, Tx_n) = \lim p(u, x_n) = 0$.

Therefore, Lemma 1 then gives $Tu = u$.

For uniqueness, let $v \in X$ a fixed point of T, hence we have either $x_0 \perp v$ or $v \perp x_0$. From the orthogonality preserving, we get $x_n \perp v$ or $v \perp x_n$, for all $n \in N$. So,

$$p(v,x_n) \le \phi\left(\max\left\{p(v,x_{n-1}), p(x_n,x_{n+1})\right\}\right);$$

then,

$$p(v,x_n) \le \phi^n\left(\max\left\{p(v,x_0), p(x_0,x_1)\right\}\right). \tag{8}$$

Using Lemma 1 and letting $n \to \infty$ in the inequality (8), we obtain: $u = v$.

Note that the inequality $p(Tx, Ty) \le \phi(p(x, y))$ implies that T is p-continuous.

Lemma 2 is proved.

Corollary. Let (X, τ) be a Hausdorff topological space with a τ-distance p. Suppose that X is p-bounded and S-complete. Let T be a p-continuous self-mapping of X such that

$$p(Tx,Ty) \le \phi\left(\max\left\{p(x,y), p(x,Tx), p(y,Ty)\right\}\right) \tag{9}$$

for all $x, y \in X$, where $\phi \in \Phi$.

Then T has a unique fixed point.

Lemma 3. *Let (X, d) be a metric space and p from $X \times X$ to $[0, +\infty)$ be a function defined by*

$$p(x,y) = e^{d(x;y)} - 1. \tag{10}$$

Then p is a τ_d-distance on X where τ_d is the metric topology.

Proof. Let (X, τ_d) be the topological space with the metric topology τ_d, let $x \in X$ and V be an arbitrary neighborhood of x, then there exists $\varepsilon > 0$ such that $B_d(x, \varepsilon) \subset V$, where

$$B_d(x,\varepsilon) = \left\{y \in X, d(x,y) < \varepsilon\right\}$$

is the open ball. It is easy to see that $B_p(x, e^\varepsilon - 1) \subset B_d(x, \varepsilon)$, indeed:

let $y \in B_p(x, e^d - 1)$, then $p(x, y) < e^\varepsilon - 1$, which implies that $e^{d(x, y)} < e^\varepsilon$, and hence $d(x, y) < \varepsilon$.

Lemma 3 is proved.

Theorem 6. *Let $(X, d, \perp)$ be an orthogonal metric space and $T: X \to X$ be a mapping such that*

$$\inf_{x \perp y, x \ne y}\left\{\max\left\{d(x,y), d(x,Tx), d(y,Ty)\right\} - d(Tx,Ty)\right\} > 0. \tag{11}$$

Then T has a unique fixed point.

Proof. Let $\alpha = inf_{x \perp y, x \ne y}\left\{\max\left\{d(x,y), d(x,Tx), d(y,Ty)\right\} - d(Tx,Ty)\right\}$,

then for all $x \ne y \in X$, with $x \perp y$, we have

$$d(Tx,Ty) \le \max\left\{d(x,y), d(x,Tx), d(y,Ty)\right\} - \alpha,$$

hence

$$e^{d(Tx,Ty)} \le ke^{\max\left\{d(x,y), d(x,Tx), d(y,Ty)\right\}}, \tag{12}$$

where $k = e^{-\alpha} < 1$.

Moreover, $x \perp y$ implies

$$p(Tx,Ty) \le k\max\left\{p(x,y), p(x,Tx), p(y,Ty)\right\}, \tag{13}$$

for all $x, y \in X$, where

$$p(x,y) = e^{d(x;y)} - 1$$

is the function mentioned in the formulation of Lemma 3, and by the inequality (13) T is an orthogonal p-continuous mapping. We also have $p(x, x) = 0$ for all $x \in X$.

Now, using Lemma 2 by taking $\phi(t) = kt$ for all $t \in [0, +\infty)$, we deduce from the inequality (13) that T has a unique fixed point.

Theorem 6 is proved.

Corollary [6]. Let T: $X \to X$ be a mapping of a bounded complete metric space (X, d) such that

$$\inf_{x\neq y}\left\{\max\left\{d(x,y),d(x,Tx),d(y,Ty)\right\}-d(Tx,Ty)\right\}>0. \tag{14}$$

Then T has a unique fixed point.

Example 3. Let $X = \{-1, 0\} \cup [1, 2]$ be equipped with the usual metric $d(x, y) = |x - y|$. Suppose that $x \perp y$ if and only if $xy \in \{-1, 0\}$; it is easy to see that $(X, \perp)$ is an O-set. Let us define T: $X \to X$ by the following conditions:

$$Tx=\begin{cases} 0, & \text{if } x\in\{-1,\ 0\}, \\ 2x, & \text{if } x\in\left[1,\ \dfrac{3}{2}\right], \\ \dfrac{x}{2}, & \text{if } x\in\left(\dfrac{3}{2},\ 2\right]. \end{cases}$$

Then T satisfies all conditions of Theorem 6 and 0 is the unique fixed point. Note that T does not satisfy all conditions (14) given by Corollary of Theorem 6; indeed,

$$\max\left\{d(0,\ 1),d(0,\ T0),d(1,\ T1)\right\}-d(T0,\ T1)=-1.$$

As applications of Theorem 6 we get a result for a new class of weakly contractive maps defined as follows.

Definition XIV. Let T: $X \to X$ be a mapping of a metric space (X, d), T will be said an orthogonal generalized E-weakly contractive map if $x \perp y$ implies

$$\begin{aligned} d(Tx,Ty)\le \max\left\{d(x,y),d(x,Tx),d(y,Ty)\right\}- \\ -\phi\left(1+\max\left\{d(x,y),d(x,Tx),d(y,Ty)\right\}\right), \end{aligned} \tag{15}$$

for all $x, y \in X$, where $\phi \in \Phi$ is a function for which the equality $\phi(1) = 0$ and inequality $\inf_{t>1} \phi(t) > 0$ hold.

Theorem 7. *Let T: $X \to X$ be an orthogonal generalized E-weakly contractive mapping of a bounded orthogonal complete metric space $(X, d, \perp)$. Then T has a unique fixed point.*

P r o o f. Let $x \neq y \in X$ and $x \perp y$, then from Definition XIV, we have

$$\begin{aligned} 0<\inf_{t>1}\phi(t)\le\phi\left(1+\max\left\{d(x,y),d(x,Tx),d(y,Ty)\right\}\right)\le \\ \le\max\left\{d(x,y),d(x,Tx),d(y,Ty)\right\}-d(Tx,Ty), \end{aligned}$$

and hence

$$\inf_{x\perp y,x\neq y}\left\{\max\left\{d(x,y),d(x,Tx),d(y,Ty)\right\}-d(Tx,Ty)\right\}>0.$$

According to Theorem 6, T has a unique fixed point in X.

Theorem 7 is proved.

Corollary [6]. Let T: $X \to X$ be an orthogonal generalized E-weakly contractive mapping of a bounded orthogonal complete metric space $(X, d, \perp)$. Then T has a unique fixed point.

Example 4. Let $X = \{0, 1, 2, 3\}$ endowed with the usual metric $d(x, y) = |x - y|$. Consider the mapping T: $X \to X$ defined as $T0 = 0 = T1$, $T2 = 3$ and $T3 = 2$.

Define a relation $\perp$ on X by

$$x \perp y \text{ if and only if } xy \le 1.$$

Then $x \perp y$ implies

$$d(Tx,Ty) \le \max\{d(x,y), d(x,Tx), d(y,Ty)\} -$$
$$-\phi(1+\max\{d(x,y), d(x,Tx), d(y,Ty)\}),$$

where $\phi \in \Phi$ is a function defined by

$$\phi(t) = \begin{cases} 0, & if\ t = 1, \\ 1, & if\ t > 1. \end{cases}$$

Therefore, all conditions of Theorem 7 are satisfied, and so T has the unique fixed point 0. On the other hand, since

$$d(T2,\ T3) = 1 > 0 = \max\{d(2,\ 3), d(2,\ T2), d(3,\ T3)\} -$$
$$-\phi(1+\max\{d(2,\ 3), d(2,\ T2), d(3,\ T3)\}),$$

the Corollary of Theorem 7 does not ensure the existence of the fixed point.

Summary and an open problem

We have established a fixed point for a new class of contractive mappings as an extension of some results (see Refs. [6, Theorem 4.2] and [3, Theorem 3]). This study was carried out only for orthogonal elements. In light of this, an open problem remains for interested researchers: whether we can generalize these results to "generalized orthogonal sets". For more details on this topic see Refs. [8, 9].

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Received 15.11.2022. Approved after reviewing 26.09.2023. Accepted 26.09.2023.
Статья поступила в редакцию 15.11.2022. Одобрена после рецензирования 26.09.2023. Принята 26.09.2023.