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THE FOURIER ANALYSIS IN INHOMOGENEOUS MEDIA

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Abstract. In the paper, the definition and basic properties of the Fourier transform (FT) are discussed. It has been shown with specific examples that integral solutions of the model inhomogeneous equation, the nonstationary Cauchy problem on an inhomogeneous shear flow, and the boundary value problem on the transformation of internal waves in the vicinity of the focus in the inhomogeneous medium can be found by FT and using its properties. The constructed Fourier integrals refuted the widely held claim that the Fourier analysis is unusable for the study of inhomogeneous media.

Keywords: Fourier analysis, Laplace transform, Cauchy problem, waves, inhomogeneous medium

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ФУРЬЕ-АНАЛИЗ В НЕОДНОРОДНЫХ СРЕДАХ

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Аннотация. В работе обсуждаются определение и основные свойства преобразования Фурье. На конкретных примерах показано, что с его помощью, а также через использование его свойств можно найти интегральные решения модельного неоднородного уравнения, нестационарной задачи Коши на неоднородном сдвиговом потоке и краевой задачи о трансформации внутренних волн в окрестности фокуса в неоднородной среде. Построенные интегралы Фурье опровергают широко распространенное утверждение, что Фурье-анализ непригоден для исследования неоднородных сред.

Ключевые слова: Фурье-анализ, преобразование Лапласа, задача Коши, волны, неоднородная среда

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Introduction

While the Fourier transformation is not the only method for solving differential equations, it is one of the most effective approaches to solving boundary-value problems, when variables are separated in a multidimensional problem. However, there is a widespread misconception about the inapplicability of Fourier analysis for inhomogeneous media. For example, the following is stated in the monograph by Whitham [1, p. 365]:

"For an inhomogeneous medium, or for nonlinear problems where the Fourier transform is not applicable...".

A similar statement is found in the book by Lighthill [2, p. 425]:

"... the need to use Fourier decomposition limits us to *homogeneous* [italicized by the author] systems usually described by equations with *constant* coefficients, so that each Fourier component (a sine wave of constant amplitude) individually can be a solution to the equations of motion."

The authors of these and many other monographs (see, for example, [3, 4]) believe that Fourier analysis can be used only in cases where the coefficients of the differential equation are constant and, conversely, it cannot be applied if these coefficients are not constant.

In this paper, we prove that Fourier analysis can be applied in problems containing differential equations with variable coefficients. Moreover, the problem can be two-dimensional and with inseparable variables, but Fourier analysis is still applicable.

Thus, the goal of this study is to expand the boundaries of the field of applicability of Fourier analysis, extending its approaches to problems in inhomogeneous media.

Definition and basic properties of the Fourier transform

The Fourier transform is defined as follows.

Forward transform:

$$\varphi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp(-ikx)\Phi(x)\,dx; \tag{1}$$

inverse transform:

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp(+ikx)\varphi(k)\,dk. \tag{2}$$

The properties of the Fourier transform can be found, for example, in monograph [1]. They are derived by differentiating with respect to the parameter or by integration by parts (see, for example, [5]). In this case, it is assumed that the function $\Phi(x)$ decreases at infinity faster than any degree of $|x|^{-1}$. Let us briefly list the properties of the transform that we will consider below.

Fourier transform of derivative function. We derive this property by integration of the formula of the forward Fourier transform (1) by parts:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp(-ikx)\frac{\partial \Phi}{\partial x}\,dx = \frac{ik}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp(-ikx)\,\Phi(x)\,dx = ik\varphi(k). \tag{3}$$

Note that the same result can be obtained by differentiating with respect to x as a parameter using Eq. (2) of the inverse Fourier transform:

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\exp(-ikx)\frac{\partial^2\Phi}{\partial x^2}dx=\frac{(ik)^2}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\exp(-ikx)\,\Phi(x)\,dx=-k^2\varphi(k). \tag{4}$$

For brevity, this property of the Fourier transform can be written as follows:

$$\Phi\to\varphi,\ \Phi_x\to ik\varphi,\ \Phi_{xx}\to -k^2\varphi. \tag{5}$$

Properties (5) are often used in problems with constant coefficients for homogeneous media.

Fourier transform of function with a linear multiplier. To represent this property, let us first integrate the transformation (2) by parts:

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\exp(+ikx)\frac{\partial\varphi}{\partial k}dk=(-ix)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\exp(+ikx)\varphi(k)\,dk=(-ix)\Phi(x). \tag{6}$$

An identical expression can be obtained by differentiating by the parameter k of relation (1). We multiply both parts of expression (6) by an imaginary unit and write it as follows:

$$x\Phi\to i\varphi_k. \tag{7}$$

Fourier transform of second derivative with a linear multiplier. We show this property by integration by parts and parametric differentiation of relation (2), repeated twice; then we obtain the following formula:

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\exp(+ikx)\frac{\partial(k^2\varphi)}{\partial k}dk=(-ix)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\exp(+ikx)k^2\varphi(k)\,dk=(ix)\frac{\partial^2\Phi}{\partial x^2}, \tag{8}$$

which we rewrite as

$$x\Phi_{xx}\to -i\left(k^2\varphi\right)_k. \tag{9}$$

One-dimensional reference equations

Finding a solution using the Fourier transform is divided into two stages. At the first stage, we construct a formal solution of the differential equation in Fourier space using operator analysis (see the previous section). At the second stage, we solve the question of the conditions under which this formally constructed solution converges. We define the integration path in the complex space and find asymptotic expressions in each sector [6, 7].

Example 1. Let us consider an inhomogeneous differential equation that arises in the analysis of wave processes in inhomogeneous plasma as well as in the study of instability in the Orr–Sommerfeld problem [6], [8, equation (1.28)]:

$$\Phi_{yyyy}+\lambda^2\left[y\Phi_{yy}+\gamma\Phi\right]=0. \tag{10}$$

Let us construct a formal solution for this example. The transform of equation (10) in Fourier space (denoted as l) has the form

$$l^4\varphi+\lambda^2\left[-i\left(l^2\varphi\right)_l+\gamma\varphi\right]=0. \tag{11}$$

Let us rewrite Eq. (11) in the following form:

$$\frac{1}{\lambda^2}l^2P-iP_l+\frac{\gamma}{l^2}P=0,\ P=l^2\varphi. \tag{12}$$

It is a homogeneous differential equation of the first order (such equations are called quadrature in the mathematical literature (see, for example, [9]).

Integrating equation (11), we obtain the following expression:

$$\varphi = \frac{1}{l^2}\exp\left(-i\frac{1}{3\lambda^2}l^3 + i\frac{\gamma}{l}\right). \tag{13}$$

The inverse Fourier transform gives the solution:

$$\varphi = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\frac{1}{l^2}\exp\left(-i\frac{1}{3\lambda^2}l^3 + i\frac{\gamma}{l} + ily\right)dl. \tag{14}$$

Next, we should perform a change of the variable $t = il$ converting the Fourier integral into the Laplace integral. It is important to note here that many consider the Laplace transform a special case of the Fourier transform (see, for example, [5, Eqs. (1.4.1), (1.4.2)]).

For convergence of integrals, we use Cauchy's theorem on the analytical function and replace the integration limits by some paths in the complex plane. We do not consider the specifics of bypassing the pole and the choice of sectors that the integration path crosses in this study, since these issues are discussed in detail in many monographs, in particular in [12], where the author relies on the Laplace transform, unlike the discussion in [5].

Thus, we obtain the following integral, which is commonly called the Laplace integral in the literature:

$$\varphi = \frac{1}{2\pi i}\int_C \frac{1}{t^2}\exp\left(\frac{1}{3\lambda^2}t^3 - \frac{\gamma}{t} + ty\right)dt. \tag{15}$$

Monograph [6] gives this integral without calculations. In our work, we show how to independently derive this formal solution with the help of the Fourier rather than Laplace transform.

Example 2. Let us consider an equation describing topographic waves on an inhomogeneous continental shelf on the *f*-plane. Monograph [19] claims that the solution of this inhomogeneous equation can be constructed using the Laplace transform, and the reader is invited to do this independently.

We in turn construct a formal solution using the Fourier transform and prove the identity of the approaches to both Fourier and Laplace transformations. Consider the equation

$$x F_{xx} + F_x + \left[\mu k - k^2 x\right]F = 0. \tag{16}$$

We introduce a dimensionless variable $\chi = kx > 0$. Eq. (16) then takes the form

$$\chi F_{\chi\chi} + F_\chi + [\mu k - \chi]F = 0. \tag{17}$$

The image of equation (17) in Fourier space with respect to the variable χ (we denote it as l) has the form

$$-i\left(l^2\Phi\right)_l + il\Phi + \mu\Phi - i\Phi_l = 0. \tag{18}$$

Let us introduce a new variable $s = il$. Eq. (18) then takes the form

$$\frac{\Phi_s}{\Phi} = \frac{\mu - s}{s^2 - 1} = \frac{\mu - 1}{2(s-1)} - \frac{\mu + 1}{2(s+1)}. \tag{19}$$

Integrating equation (19) and performing the inverse Fourier transform, we obtain the following formal integral:

$$F(x) = \frac{1}{\sqrt{2\pi i}}\int_C \frac{(s-1)^{(\mu-1)/2}}{(s+1)^{(\mu+1)/2}}\exp(skx)\,ds. \tag{20}$$

The integral constructed here coincides with Eq. (25.20) in monograph [19] up to a multiplier; the monograph also presents analysis of integral (20) with the choice of integration paths.

Results obtained in the section "One-dimensional reference equations". Following the approach outlined in monograph [5], we can change the variable, and then the properties of the Fourier transform are transferred to the properties of the Laplace transform. Thus, according to the statement made in [5], the Fourier transform is a kind of basic transformation, from which other transformations, for example, Laplace and Mellin types, follow. A similar approach is followed by the authors of monograph [9], who constructed fundamental solutions to the thermal conductivity operator, the Laplace and Helmholtz operators, as well as the wave operator in terms of the Fourier transform.

Thus, there is no fundamental difference between Fourier analysis and the Laplace transform in one-dimensional inhomogeneous media. It can be assumed that the solution is constructed in terms of the Laplace transform, but it can also be argued that the solution is derived in terms of the Fourier transform and Cauchy's theorem. Following [5] from now on, we adhere to the second approach.

Unsteady Cauchy problem for Rossby waves in zonal current

As the first example of a two-dimensional Fourier transform in inhomogeneous media, consider the unsteady Cauchy problem for Rossby waves. Yamagata solved this problem in 1976 using convective coordinates [20]. Convective coordinates are a common way for an operator of the type $\partial t + U(y)\partial x$ for the case of a linear velocity profile $U(y) = U_y y$, convective coordinates transform an inhomogeneous differential equation into a homogeneous one, and then the Fourier transform is applied over spatial convective coordinates (two-dimensional transform). Next, an unsteady differential equation with respect to t is obtained.

It is fundamentally important that there is no point in doing the Laplace transform with respect to the variable t, as is customary in some mathematical groups (see, for example, [15]). It is easier to solve the differential equation with respect to t explicitly than to perform an additional transformation. By solving the differential equation with respect to t and taking the inverse Fourier transform in convective coordinates, we can convert convective variables to ordinary ones and get a solution in the form of a two-dimensional Fourier transform.

Next, we can find a solution to an inhomogeneous differential equation and, repeating the calculations, obtain the Yamagata solution, but we propose a different approach. We will not adopt the convective coordinates to eliminate the inhomogeneity of the differential equation to then perform the Fourier transform. We will immediately apply the Fourier transform for an inhomogeneous differential equation using its properties. Thus, on the one hand, we will significantly reduce the number of operations, and on the other hand, we will arrive at a known result and confirm the correctness of mathematical calculations. We will demonstrate this approach for the problem solved above, but we will solve it by a new, shorter technique, which allows to immediately find the Fourier transform of an inhomogeneous differential equation.

Example 3. The linear Cauchy problem for non-divergent barotropic Rossby waves in zonal shear flow is considered in [20], and its generalization to the case of divergent waves is considered in [18]:

$$\left(\partial_t + U_y y \partial_x\right)\left[\Psi_{xx} + \Psi_{yy}\right] + \beta\Psi_x = 0, \tag{21}$$

where Ψ is the function of current; β is the classical parameter, $\beta = \dfrac{df}{dy}$ ($f = 2\Omega\sin\varphi$, Ω is the angular velocity of the Earth's rotation, φ is the latitude); the x axis is directed to the east, the y axis is to the north.

Let there be inhomogeneous zonal shear flow $U(y) = U_y y$, where $U_y = \text{const}$. Let us perform a two-dimensional Fourier transform for inhomogeneous differential equation (22) with respect to two spatial variables x and y (without adopting the convective variables):

$$\Psi\left(x, y, t\right) = \frac{1}{2\pi} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \varphi\left(k, l, t\right) \exp\left[+i\left(kx + ly\right)\right] dk\, dl. \tag{22}$$

Then Eq. (21) takes the following form in Fourier space

$$\left[\left(-k^2-l^2\right)\varphi\right]_t - U_y k\left[\left(-k^2-l^2\right)\varphi\right]_l + i\beta k\varphi = 0, \tag{23}$$

where the subscripts denote partial derivatives.

This equation is homogeneous, contains only the first partial derivatives and is easily solved.

Let us rewrite Eq. (23) in the following form:

$$P_t - U_y k P_l - \frac{i\beta k}{k^2+l^2}P = 0,\ P \equiv \left(k^2+l^2\right)\varphi. \tag{24}$$

Performing a substitution of variables ($\tau = t$, $l' = l + kU_y t$), we obtain the following equation:

$$P_\tau - \frac{i\beta k P}{k^2 + \left(l' - kU_y\tau\right)^2} = 0. \tag{25}$$

Eq. (25) can be integrated explicitly (the exponent of the arctangent), and then the final solution has the form of a double Fourier integral:

$$\Psi(x,y,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g_1(k,l)\frac{k^2+l^2}{k^2-\left(l-kU_y t\right)^2}\times$$
$$\times\exp\left[-i\frac{\beta}{kU_y}\left\{\arctan\left(\frac{l}{k}\right)-\arctan\left(\frac{l}{k}-U_y t\right)\right\}\right]\times \tag{26}$$
$$\times\exp\left[+i\left(kx+\left(l-kU_y t\right)y\right)\right]dk\,dl,$$

where the solution is normalized to the initial condition

$$g_1(k,l) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\Psi(x,y,t=0)\exp\left[-i\left(kx+ly\right)\right]dx\,dy. \tag{27}$$

Analysis of the double Fourier integral by the stationary phase method and the construction of wave packet trajectories can be found in [20]. It is important to note that these studies do not rely on the assumption that the time variable must be large.

Unsteady Cauchy problem for Rossby waves in meridional current

As a second example, consider the unsteady Cauchy problem for Rossby waves in meridional current. A solution to this problem using convective coordinates can be found in [20].

Example 4. This is the case of a linear velocity profile of meridional current. The linear Cauchy problem for divergent barotropic Rossby waves has the following form [20]:

$$\left(\partial_t + V_x x\partial_y\right)\left[\Psi_{xx}+\Psi_{yy}\right]+\beta\Psi_x = 0, \tag{28}$$

where β is the classical parameter; the x axis is directed to the east, the y axis is directed to the north.

There is inhomogeneous meridional shear flow $V(x) = V_x x$, where $V_x = const.$ As before (see Example 3), we perform a two-dimensional Fourier transform for inhomogeneous differential equation (28) with respect to two spatial variables x and y. Then Eq. (28) takes the following form in Fourier space

$$\left[\left(-k^2-l^2\right)\varphi\right]_t - U_x l\left[\left(-k^2-l^2\right)\varphi\right]_k + i\beta k\varphi = 0. \tag{29}$$

This equation is homogeneous, contains only the first partial derivatives and is easily solved. Let us rewrite it in the following form:

$$P_t - U_x l P_k - \frac{i\beta k}{k^2 + l^2} P = 0, \; P \equiv \left(k^2 + l^2\right)\varphi. \tag{30}$$

Performing a change of variables ($\tau = t$, $k' = k + lU_x t$), we obtain the following equation:

$$P_\tau - \frac{i\beta\left(k' - lU_x\tau\right)P}{\left(k' - lU_x\tau\right)^2 + l^2} = 0. \tag{31}$$

Eq. (31) can be integrated explicitly (the exponent of the logarithm), and then the final solution has the form of a double Fourier integral:

$$\Psi(x,y,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g_2(k,l)\frac{k^2 + l^2}{\left(k - lU_x t\right)^2 + l^2}\exp\left[i\frac{\beta}{2lU_x}\ln\left\{\frac{\left(k - lU_x t\right)^2 + l^2}{k^2 + l^2}\right\}\right]\times$$
$$\times\exp\left[+i\left(\left(k - lU_x t\right)x + ly\right)\right]dk\,dl, \tag{32}$$

where the solution is normalized to the initial condition

$$g_2(k,l) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\Psi(x,y,t=0)\exp\left[-i\left(kx + ly\right)\right]dx\,dy. \tag{33}$$

Analysis of the double Fourier integral by the stationary phase method and the construction of wave packet trajectories can be found in [20].

The considered examples are simple in the sense that the obtained double integrals are already known. The novelty of our solution lies in the fact that the solution is formally constructed by a direct Fourier transform of an inhomogeneous differential equation without involving convective coordinates.

Let us now move on to a more complex problem, where the solution in the form of the Fourier integral of the boundary-value problem was not previously known, but it was considered in terms of special functions with respect to complex variables. The very procedure of constructing a certain complex variable for a special hypergeometric function and integration along a certain circle in a complex space suggests that there must be a way to obtain this solution in terms of the direct Fourier transform of the original inhomogeneous differential equation with inseparable variables.

Reference equation for a two-dimensional inhomogeneous medium. Abnormal focusing of internal waves

In the examples discussed above, the solution was sought in the form of a two-dimensional Fourier integral, while the inhomogeneity of the external field (the velocity field of the background flow, or topography) was one-dimensional.

Now let us consider a more complex example of a problem with two-dimensional inhomogeneity of the external field.

Example 5. The theory of anomalous focusing of internal waves in a two-dimensional inhomogeneous fluid introduces the following reference equation of elliptical-hyperbolic type for vertical displacement in the vicinity of the focus [13, Eq. (2.5)]:

$$\Psi_{zz} + \left(\frac{y}{L_y} + \frac{z^2}{L_z^2}\right)\Psi_{yy} + \frac{2}{L_y}\Psi_y = 0, \tag{34}$$

where Ψ is the function of current; (x, y, z) is the rectangular coordinate system; L_y, L_z are the lengths of inhomogeneities along the y and z axes.

We will search for solutions localized in a small neighborhood of a certain level along the vertical coordinate and exponentially attenuating outside this level; here, for the case of internal waves, the following notations are introduced [13]:

$$\frac{1}{L_y} = 2\nabla_y \ln\Omega, \; \frac{1}{L_z^2} = \frac{\nabla_z^2\Omega}{\Omega} - \frac{\nabla_z^2 N}{N}, \tag{35}$$

where $\Omega = \omega - kU$ (ω is the frequency, k is the zonal wavenumber, $U(z,y)$ is the inhomogeneous horizontal background shear flow); $N^2(z) = -g\frac{d}{dz}\ln\rho_0(z)$ ($\rho_0(z)$ is the density).

The value of all derivatives is taken at the focal point. Since equation (34) is invariant with respect to the scale transformation $z = az'$, $y = a^2y'$, a certain self-similar variable is introduced. The solution is constructed as a summation of all partial solutions with respect to hypergeometric functions of a complex argument. The procedure for constructing this complex variable is not entirely clear. It is also not entirely clear what functions are considered in [13], how these functions appear, what their physical nature is, and what primary and secondary quantization mean in construction of asymptotic forms of the solution. Note that the asymptotic forms of the two-dimensional function are constructed as one-dimensional only on the waveguide axis.

To interpret these solutions in terms of special functions of complex arguments on the one hand and to represent these classes of solutions using the classical Fourier transform and its properties (presented above) on the other hand, we will independently construct a solution in integral form, find its two-dimensional asymptotic forms and show what primary and secondary quantization mean in terms of the classical Sturm–Liouville problem. To do this, we can use the well-known integral representations of the hypergeometric function as a basis, and then the approach to finding a solution will become more transparent. In a sense, we are using the integral representation as a starting point, but it is better to take steps in the opposite direction to search for the solution.

The solution to equation (34) is sought in the form of a Fourier integral. First we confine ourselves to the upper half of the integral:

$$\Psi(k,y,z,\omega) = \int_0^\infty G(k,l,z,\omega)\exp(ily)\,dl. \tag{36}$$

In fact, a reasonable question arises whether to take the whole integral or only the upper (or lower) part of it. We will further discuss this problem in the final section of the paper.

We use the properties of the Fourier transform again:

$$\Psi \to G,\ \Psi_y \to ilG,\ \Psi_{yy} \to -l^2G,\ y\Psi_{yy} \to -i\left(l^2G\right)_l. \tag{37}$$

The first three formulas in this equation are the properties of the Fourier transform of the derivative, which are widely known. The latter formula is a special case of equality (21) in monograph [9]. Despite the popularity of this formula, it is not used in applied problems. Our work focuses specifically on the practical application of this last formula from Eq. (37).

Substituting integral (36) into equation (34) and taking into account (37), we obtain the following equation for the Fourier transform G:

$$G_{zz} - \frac{l^2z^2}{L_z^2}G - i\frac{l^2}{L_y}G_l = 0. \tag{38}$$

Equality (38) is not an equation with separable variables. To convert it to such form, let us perform the following variable substitution

$$(z,l) \to (\eta,\varphi),$$

where

$$\eta = \frac{z\,l^{1/2}}{L_z^{1/2}},\ \varphi = l. \tag{39}$$

The Jacobian of such a substitution has the form

$$\frac{\partial(\eta,\varphi)}{\partial(z,l)} = l^{1/2}. \tag{40}$$

It should be noted that equations (39) and (40) contain $l^{1/2}$. Technically, this specific fact allows us to consider only one of the parts of the Fourier integral. For simplicity, we first chose the upper, positive part of the integration to resolve the question related to the square root.

This however raises the question of why such a variable substitution should be chosen. The answer is contained in [17], where the solution is constructed in the WKB approximation. In fact, any reasoning about self-similarity turns out to be superfluous, since in a certain sense the entire self-similarity of the solution is reduced to a simple substitution of variables of the form (39).

Equality (38) in terms of new variables (η,φ), takes the form of an equation with separable variables:

$$G_{\eta\eta} - \eta^2 G - i\frac{\eta L_z}{2L_y}G_\eta - i\frac{\varphi L_z}{L_y}G_\varphi = 0. \tag{41}$$

In this case, we search for a solution with separable variables:

$$G(\eta,\varphi) = H(\eta)F(\varphi). \tag{42}$$

We obtain the following equation for the function $H(\eta)$:

$$H_{\eta\eta} - i\frac{\eta L_z}{2L_y}H_\eta - \left(\eta^2 + \mu_0\right)H = 0, \tag{43}$$

where μ_0 is the separation constant.

Next, the term with the first derivative in equation (43) is removed by the following substitution:

$$H(\eta) = P(\eta)\exp\left(i\frac{L_z}{8L_y}\eta^2\right). \tag{44}$$

We obtain the following equation for the function $P(\eta)$:

$$P_{\eta\eta} + \left[-\eta^2\left(1 - \frac{L_z^2}{16L_y^2}\right) - \mu_0 + i\frac{L_z}{4L_y}\right]P = 0. \tag{45}$$

Recall that we are searching for solutions localized in the vicinity of the level $z = 0$. Analysis of equation (45) allows to conclude that the coefficient at η^2 must be positive, so we obtain the following condition for the existence of localized solutions:

$$\left(1 - \frac{L_z^2}{16L_y^2}\right) > 0 \Leftrightarrow 0 < |L_z| < 4|L_y|. \tag{46}$$

Condition (46) means that the branches of the parabola bounding the inner area of transparency from the outer area of shadow, must be practically parallel to each other. Otherwise, the vertical mode does not form and the wave does not approach the critical point for an infinitely long time. It is important to note that if condition (46) is not satisfied, then other modes of solution transformation are formally possible. There is no question of any uniqueness of the solution here.

Evaluation of the parameters for internal waves shows that if we take the scales adopted by the authors of [13], we obtain a very good difference in these values ($L_z < 4\ L_y$), so the concept of a parabolic trap is valid from a physical standpoint.

Let us define the quantum values of the separation variable μ_0 [10, 16]:

$$-(2m+1) = \left(\mu_0 - i\frac{L_z}{4L_y}\right) / \left(1 - \frac{L_z^2}{16L_y^2}\right)^{1/2},\ m = 0, 1, 2, \dots . \tag{47}$$

From here, we can find the eigenvalues

$$\mu_0 = \frac{L_z}{L_y}\left[\frac{1}{4}i - \frac{\delta}{2}\left(m+\frac{1}{2}\right)\right];\ \delta \equiv \left(\frac{16L_y^2}{L_z^2}-1\right)^{1/2},\ m = 0,1,2,... \tag{48}$$

and eigenfunctions

$$P(\eta) = \left[\sum_{m=0}^{\infty} H_m\left(\eta\left[1-\frac{L_z^2}{16L_y^2}\right]^{1/4}\right)\right]\exp\left[-\frac{\eta^2}{2}\left(1-\frac{L_z^2}{16L_y^2}\right)^{1/2}\right],\ m = 0,1,2,..., \tag{49}$$

where H_m are Hermite polynomials.

Let us now define the second factor $F(\varphi)$ in solution (42). We obtain the following equation from Eq. (41):

$$-i\frac{\varphi L_z}{L_y}F_\varphi + \mu_0 F = 0. \tag{50}$$

The solution of equation (50) has the following form:

$$F(\varphi) = \varphi^\mu,\ \mu \equiv -i\mu_0\frac{L_y}{L_z}. \tag{51}$$

Finally, we obtain the following eigenvalues:

$$\mu = \frac{1}{4} + i\frac{\delta}{2}\left(m+\frac{1}{2}\right). \tag{52}$$

Substituting all the found composite solutions into the initial integral (36), we find the eigenfunctions:

$$\Psi(k,y,z,\omega) = A(k,\omega)\sum_{m=0}^{\infty}\int_0^{\infty} l^\mu\left[H_m\left(\frac{z\,l^{1/2}}{L_z^{1/2}} - \left[1-\frac{L_z^2}{16L_y^2}\right]^{1/4}\right)\right]\times$$
$$\times\exp\left[-\frac{z^2 l}{2L_z}\left(1-\frac{L_z^2}{16L_y^2}\right)^{1/2}\right]\cdot\exp\left[il\left(y+\frac{z^2}{8L_y}\right)\right]dl, \tag{53}$$

where $A(k,\omega)$ is some constant that determines the spectral density of the initial state.

Further, the obtained eigenfunctions (53) can be reduced by simple transformations to a degenerate hypergeometric function with respect to some complex argument. Note that it is the integral notation (53) that is preferred for finding the asymptotic forms of eigenfunctions. Despite the fact that the constructed eigenfunctions (53) express the dependence on two physical variables (z and y), the integral for the eigenfunctions is one-dimensional, which makes it possible to use the stationary phase method [16].

Let us write the imaginary part of the integral (53) in the following form:

$$\exp\left[il\left(y+\frac{z^2}{8L_y}\right) + i\frac{\delta}{2}\left(m+\frac{1}{2}\right)\ln l\right]. \tag{54}$$

If we differentiate this expression by the variable l and equate the expression in square brackets to zero, we obtain the equation for the point l_c:

$$y + \frac{z^2}{8L_y} = -\frac{\delta}{2l_c}\left(m+\frac{1}{2}\right). \tag{55}$$

Let us rewrite this relation in the following form:

$$l_c = -\frac{\delta\left(m+\frac{1}{2}\right)}{2\left(y+\frac{z^2}{8L_y}\right)}. \tag{56}$$

The resulting expression (56) is a kind of generalization of the short-wave WKB asymptotic form of the dispersion relation $l_c = y^{-1}$. Then the second derivative of the phase with respect to the wavenumber is proportional to l_c^{-2}, and, therefore, the inverse first power root of this derivative is proportional to $l^1{}_c$.

The asymptotic form of eigenfunctions in the vicinity of the critical point is as follows:

$$\Psi_1(k,y,z,\omega) = A(k,\omega)\sum_{m=0}^{\infty} l_c^{\mu+1}\left[H_m\left(\frac{z\,l_c^{1/2}}{L_z^{1/2}}\left[1-\frac{L_z^2}{16L_y^2}\right]^{1/4}\right)\right]\times$$
$$\times\exp\left[-\frac{z^2 l_c}{2L_z}\left(1-\frac{L_z^2}{16L_y^2}\right)^{1/2}\right]\cdot\exp\left[i\,\frac{\delta}{2}\left(m+\frac{1}{2}\right)\right]. \tag{57}$$

Analysis of this equality allows us to conclude that the asymptotic form of the solution of the reference equation exactly coincides with the WKB solution [17], expressed as a vertical mode in the form of Hermite polynomials majored by a Gaussian function, and gives the classical degree of 5/4 for the amplitude of the vertical velocity. If the authors of [13] describe a certain mode, then we are certain that this is not their vertical mode in the form of a WKB solution along a vertical coordinate, but a completely different one, which is constructed in [17].

The solutions we constructed are not functions with respect to the variables *z*, *y*, but rather to some curvilinear variables taking the following form:

$$(y,z)\to\left(\left(y+\frac{z^2}{8L_y}\right),\quad \frac{z}{\left(y+\frac{z^2}{8L_y}\right)^{1/2}}\right). \tag{58}$$

Thus, in a sense, there is a curvature of space in the vicinity of the focal point. However, all this "curvilinearity" was also observed in the solution of the problem in the WKB approximation, where formally the following variable substitution took place:

$$(y,z)\to\left(y,\frac{z}{\sqrt{y}}\right).$$

Therefore, by and large, the asymptotic forms of one-dimensional integrals do not give any qualitatively new results other than WKB solutions, with the exception of condition (47), which is satisfied with a large margin.

Reduction of the Fourier integral to a hypergeometric function of a complex variable. To compare our solution with the solution obtained by Erokhin and Sagdeev [13], we rewrite the eigenfunctions (54) in the following form:

$$\Psi_m(k,y,z,\omega) = \int_0^{\infty} l^{\mu} H_m\left(\frac{zl^{1/2}}{2L_y^{1/2}}\delta^{1/2}\right)\exp\left[-\frac{z^2 l}{8L_y}\delta\right]\cdot\exp\left[il\left(y+\frac{z^2}{8L_y}\right)\right]dl. \tag{59}$$

Next, we perform the substitution of variables ($l \to x$), and the argument

$$\left(\frac{z\, l^{1/2}}{2L_y^{1/2}}\delta^{1/2}\right)$$

of the Hermite polynomial is taken as a new variable

$$x = \frac{z l^{1/2}\delta^{1/2}}{2L_y^{1/2}}. \tag{60}$$

It follows from this that

$$\Psi_m \propto \int_0^\infty \exp\left[-2ax^2\right]\frac{x^{2\mu+1}}{z^{2\mu+2}}H_m(x)\,dx. \tag{61}$$

The complex variable $2a$ appeared in Eq. (60), depending on two spatial physical variables, z and y:

$$2a = \frac{1}{2} - i\frac{1}{2\delta z^2}\left(z^2 + 8yL_y\right). \tag{62}$$

We solved the two-dimensional problem in terms of a one-dimensional integral, but only with respect to a complex argument. Apparently,

$$\frac{1}{2a} = \frac{2\delta z^2}{\delta z^2 - i\left(z^2 + 8\,yL_y\right)} \equiv \tau^*, \tag{63}$$

where τ is the complex variable from the study by Erokhin and Sagdeev [13] (the asterisk corresponds to complex conjugation).

The integral representation of the hypergeometric function in terms of Hermite polynomials has the following form [11, Eqs. 7.37, 7.38]:

$$F\left(-n; \frac{\nu+1}{2}; \frac{1}{2}; \frac{1}{2a}\right) \sim \int_0^\infty \exp\left[-2ax^2\right] x^\nu H_{2n}(x)dx, \tag{64}$$

where $\mathrm{Re}\,a > 0$, $\mathrm{Re}\,\nu > -1$;

in addition [11, Eqs. 7.376.3]:

$$F\left(-n; \frac{\nu}{2}+1; \frac{3}{2}; \frac{1}{2a}\right) \sim \int_0^\infty \exp\left[-2ax^2\right] x^\nu H_{2n+1}(x)dx, \tag{65}$$

where $\mathrm{Re}\,a > 0$, $\mathrm{Re}\,\nu > -2$.

Taking into account the eigenvalues (48), we find:

$$\nu = 2\mu + 1 = \frac{3}{2} + i\delta\left(m + \frac{1}{2}\right). \tag{66}$$

Consequently, the constructed solutions are regular, and the integrals converge. Similarity with the solution from Erokhin and Sagdeev [13] was achieved in three of the four parameters. Let us determine the last parameter of the hypergeometric function:

$$\frac{\nu}{2} + 1 = \frac{7}{4} + i\frac{\delta}{2}\left(m + \frac{1}{2}\right) \equiv \gamma^*, \tag{67}$$

where γ is the quantum parameter from [1, Eq. (2.7)].

Similarly, we find that

$$\frac{\nu+1}{2} = \frac{5}{4} + i\frac{\delta}{2}\left(m + \frac{1}{2}\right). \tag{68}$$

Thus, we obtained a complete agreement of our results with [13]. If we take into account the second part of the Fourier integral for negative wave numbers, then by substitution of the variable it can be reduced to an integral with respect to positive wavenumbers. But then the imaginary unit ($i \to -i$ 1) will be replaced in the studied integral, and this will lead to the appearance of the second part of the solution, where complex conjugate τ^* and γ^* will appear instead of τ and γ.

Thus, the general solution of the problem is the sum of solutions with respect to τ and τ^*, which is physically equivalent to the sum of incident and reflected waves. This means that mathematically there is no prohibition on reflection and the hypothesis of infinite focusing is greatly exaggerated.

Discussion and conclusions

This study provides basic information on the operator method of Fourier transformation, which is necessary for practical solution of specific physical problems in inhomogeneous media. The main properties are formulated by two approaches:

integration by parts, which implies attenuation of functions at infinity;

parametric differentiation of the forward or inverse Fourier transform.

Using five specific examples, we established how Fourier analysis works in inhomogeneous media. In the first four examples, formal integral solutions are constructed, since their further analysis is well known and the reader can consider the references to further explore this. Notice that these integrals (see examples 1 and 2) are typically given without derivation and the reader is invited to independently obtain this derivation using the Laplace transform. In our work, we constructed integral solutions using the Fourier transform and Cauchy's theorem, showing their equivalence with the Laplace transform in one-dimensional inhomogeneous problems.

Examples 2, 3 and 4 consider a two-dimensional problem in which the inhomogeneity of the medium is one-dimensional linear in nature. In Example 2, the solution can be obtained in two ways: in terms of the Fourier transform and in terms of the Laplace transform. In Example 5, we performed a complete analysis of the boundary-value problem. We constructed the Fourier integral, found its two-dimensional asymptotic forms using the stationary phase method and the properties of a parabolic quantum oscillator, and also identified the Fourier integral found, reducing it to a well-known degenerate hypergeometric function with respect to a complex argument. Thus, we proved that the statement about the inefficiency of Fourier analysis in inhomogeneous media is erroneous.

Therefore, in terms of the Fourier integral, we analytically proved the identity of the solution of the reference equation for vertical focusing of a monochromatic wave in the vicinity of the focus with the solution of the reference equation in terms of a degenerate hypergeometric function with respect to a complex variable obtained in previous studies. This mathematical solution is also successfully used in problems of magnetohydrodynamic instability and in the description of internal gravitational waves in two-dimensional inhomogeneous fluid [7, 13].

It is established that the issue of wave absorption in the focal zone is ambiguous and therefore both passage and reflection from a singularity can be observed. Specific estimates for typical parameters of oceanic gradients of hydrophysical density and velocity fields show that localization and, as a rule, amplification of wave movements are quite feasible and take the form of highly localized spatial vortex structures.

These aspects should be taken into account in studies of geophysical fields, in particular when analyzing mesoscale vortex dynamics in the ocean.

The analytical method described in these five examples can be used to solve other problems of mathematical physics.

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