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THE STOKES PROBLEM FOR AN ELLIPTIC CONTOUR

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Abstract. The paper considers the problem on small harmonic oscillations of an elliptical contour immersed in the incompressible viscous fluid. Analytical and asymptotic methods for solving this problem have been proposed. The results obtained in the numerical implementation of the analytical method and the results of asymptotic solutions were compared. The possibilities of describing the solution in almost the entire range of values of the dimensionless viscosity parameter by joint application of the proposed methods were shown.

Keywords: Stokes problem, viscous incompressible fluid, solid body vibrations, elliptical cylinder

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ЗАДАЧА СТОКСА ДЛЯ ЭЛЛИПТИЧЕСКОГО КОНТУРА

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Аннотация. В работе рассмотрена задача о малых гармонических колебаниях эллиптического контура, погруженного в несжимаемую вязкую жидкость. Предложены аналитический и асимптотические методы решения этой задачи. Приведено сопоставление результатов, полученных при численной реализации аналитического метода, с результатами асимптотических решений. Показано, что совместное использование предложенных методов позволяет описать решение почти во всем диапазоне значений безразмерного параметра вязкости.

Ключевые слова: задача Стокса, вязкая несжимаемая жидкость, колебания твердого тела, эллиптический цилиндр

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Introduction

Computations of hydrodynamic forces acting on solid and elastic structures oscillating in viscous incompressible fluid are important for diverse applications, for example, nanotechnology, viscosity measurements, hydromechanics of marine structures.

Since structures typically have a complex geometric shape, computational experiments conducted by grid-based numerical simulation are used to solve such problems, implemented in well-known commercial CAE packages (for example, Ansys Fluent, STAR-SSM+, etc.). The computational approach is universal, but very expensive and time-consuming.

In view of these circumstances, it is important to be able to solve relatively simple reference problems for which explicit, asymptotic or numerical solutions can be constructed. The solutions obtained can be used as a first and often good approximation for more problems more difficult to formulate.

To date, the only plane problem (apart from the model problem of vibrations in a plane) allowing for an exact analytical solution is the problem of small oscillations in a circular cylinder in an incompressible viscous fluid at rest, whose solution was obtained by Stokes in 1851.

Such a simplified model has long been used to solve hydrodynamic problems, for example, in the theory of atomic force microscopy, and still remains popular. However, the estimate of the hydrodynamic reaction of structures with a complex geometric shape is clearly very approximate if it is based on Stokes's results for a circular cylinder.

In this paper, we consider the Stokes problem for an elliptical contour making small harmonic oscillations in incompressible viscous fluid, proposing a combination of methods to describe its solution in almost the entire range of values of dimensionless viscosity.

Problem statement

Let an arbitrary planar contour immersed in incompressible viscous fluid with kinematic viscosity ν and density ρ make small harmonic oscillations in its plane with a given frequency ω. The oscillation amplitude is assumed to be much smaller than the size of the contour.

The equations of motion are written in dimensionless form using the characteristic size of the contour L as a unit of length, the quantity ωL is divided by the amplitudes of velocities v_x, v_y of the fluid and the forced oscillations of the contour u_0, $\rho\omega^2 L^2$ is divided by the pressure, $\rho\omega^2 L^3$ is divided by the hydrodynamic force F_x.

Under the above assumptions, hydrodynamic equations of an incompressible viscous fluid take the form [1]:

$$\begin{gathered}\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial y^2}-\beta\frac{\partial p}{\partial x}+i\beta v_x=0,\\ \frac{\partial^2 v_y}{\partial x^2}+\frac{\partial^2 v_y}{\partial y^2}-\beta\frac{\partial p}{\partial y}+i\beta v_y=0,\\ \frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}=0,\end{gathered}\tag{1}$$

where $\beta = \omega L^2/\nu$ is a dimensionless parameter.

No-slip conditions are imposed on the contour line Γ:

$$v_n\big|_\Gamma = u_n,\;\; v_\tau\big|_\Gamma = u_\tau,\tag{2}$$

where (v_n, v_τ), (u_n, u_τ) are the normal and tangential components of the velocity vectors of the fluid and the contour, respectively.

Oscillation-induced perturbations of the velocity field decay with distance from the contour Γ:

$$v_x \to 0,\; v_y \to 0,\; \sqrt{x^2+y^2} \to \infty.\tag{3}$$

If the components of the velocity vector are represented in terms of two scalar functions φ and ψ, referred to as potentials for brevity [2], namely,

$$v_x = \frac{\partial\varphi}{\partial x} + \frac{\partial\psi}{\partial y},\ v_y = \frac{\partial\varphi}{\partial y} - \frac{\partial\psi}{\partial x},$$

then system of equations (1) and boundary conditions (2), (3) can be written as

$$\begin{cases} \Delta\varphi = 0, \\ \Delta\psi + i\beta\psi = 0, \\ p = i\varphi; \end{cases} \tag{4}$$

$$\begin{cases} \dfrac{\partial\varphi}{\partial n} + \dfrac{\partial\psi}{\partial\tau} = u_n, \\ \dfrac{\partial\varphi}{\partial\tau} - \dfrac{\partial\psi}{\partial n} = u_\tau, \\ \varphi, \psi \to 0, \sqrt{x^2 + y^2} \to \infty. \end{cases} \tag{5}$$

As established in [3], the hydrodynamic force F_x acting on a plane contour with a normal **n** = (n_x, n_y) making small harmonic oscillations along the *Ox* axis is determined in terms of the potentials by the formula

$$F_x = -i\int_{\tilde{A}} \varphi n_x dl - i\int_{\tilde{A}} \psi n_y dl. \tag{6}$$

The problem formulated is of both theoretical and practical interest in various fields, such as microsystem design (scanning probe microscopy [4–6]), instrumentation for viscosity measurements [7], hydromechanical problems of offshore structures [8]. In practice, the values of the dimensionless parameter β can vary in a wide range: from 10^{-3} to 10^5.

The exact solution of the problem of oscillations in a circular cylinder immersed in incompressible viscous fluid at rest, as well as the formula for calculating the hydrodynamic drag of the cylinder, were obtained by George Stokes in 1851 [9]. This result, which may be cumbersome to access, was subsequently reproduced in [10] and confirmed experimentally (see, for example, [11, 12]).

By analogy with this classical problem, we define the problem of vibrations in an arbitrary plane contour placed in infinite incompressible fluid as the Stokes problem for this contour.

Stokes problem for a circular cylinder

We present the solutions of the problem obtained by the potential method from monograph [2]. The solution for the case of incompressible fluid has the following form in polar coordinates *r*, θ:

$$\begin{aligned} \varphi(r,\theta) &= u_0 \frac{\left(H(1,\beta)-1\right)}{\left(H(1,\beta)+1\right)} \frac{1}{r} \cos\theta, \\ \psi(r,\theta) &= u_0 \frac{2H(r,\beta)}{\left(H(1,\beta)+1\right)} \sin\theta, \\ F_x(\beta) &= -i\pi u_0 \frac{3H(1,\beta)-1}{1+H(1,\beta)}, \end{aligned} \tag{7}$$

introducing the function

$$H(r,\beta) = \frac{H_1^{(1)}\left(r\sqrt{i\beta}\right)}{\sqrt{i\beta}\, H_1'^{(1)}\left(r\sqrt{i\beta}\right)\Big|_{r=1}}. \tag{8}$$

For large values of β, it follows from Eq. (7) for the hydrodynamic force that

$$F_x(\beta) \underset{\beta\to\infty}{\sim} i\pi u_0 \left[1 + 2(1+i)\sqrt{\frac{2}{\beta}}\right]. \tag{9}$$

Eq. (9) is often called the Stokes formula [10–12].

Evidently, if $\beta \to \infty$, $F_x(\beta) \to i\pi u_0$, which corresponds to representations of the 'added' mass of ideal incompressible fluid.

At low β, the potentials have the form

$$\begin{aligned} \varphi(r,\theta) &\underset{\beta\to 0}{\sim} \frac{4iu_0}{\beta\left(2\gamma - \frac{i\pi}{2} - 2\ln 2 + \ln\beta\right)} \frac{1}{r}\cos\theta, \\ \psi(r,\theta) &\underset{\beta\to 0}{\sim} \frac{4iu_0}{\beta\left(2\gamma - \frac{i\pi}{2} - 2\ln 2 + \ln\beta\right)} \frac{1}{r}\sin\theta, \end{aligned} \tag{10}$$

and, in accordance with Eq. (6),

$$F_x \underset{\beta\to 0}{\sim} \frac{8\pi u_0}{\beta\left(2\gamma - \frac{i\pi}{2} - 2\ln 2 + \ln\beta\right)}. \tag{11}$$

(here γ is Euler's constant)

Methods for solving the Stokes problem for an elliptic cylinder

Analytical solution. Some asymptotic results were obtained in [13] for an elliptic contour, and a formal solution was constructed in [14].

Thus, we take the major semiaxis of the ellipse ($L = a$) as the length scale. The ellipse equation in Cartesian (dimensionless) coordinates has the form

$$\Gamma: x^2 + \frac{y^2}{\varepsilon^2} = 1, \quad \left(\varepsilon = \frac{b}{a}\right).$$

We introduce elliptical coordinates

$$x = h\,\mathrm{ch}\,\xi\cos\eta, \ y = h\,\mathrm{sh}\,\xi\sin\eta.$$

where $h = \sqrt{1-\varepsilon^2}$ is half the distance between the focal points,

$$\xi \in [\xi_0, +\infty), \ \eta \in [0, 2\pi].$$

Then the ellipse equation is written as

$$\xi = \xi_0, \ \left(\mathrm{ch}\,\xi_0 = \frac{1}{\sqrt{1-\varepsilon^2}}, \mathrm{sh}\,\xi_0 = \frac{\varepsilon}{\sqrt{1-\varepsilon^2}}\right).$$

The unit normal **n** and tangential τ vectors to the contour of the ellipse are expressed as

$$\mathbf{n} = \frac{(\varepsilon\cos\eta, \sin\eta)}{\sqrt{\sin^2\eta + \varepsilon^2\cos^2\eta}}, \quad \boldsymbol{\tau} = \frac{(-\sin\eta, \varepsilon\cos\eta)}{\sqrt{\sin^2\eta + \varepsilon^2\cos^2\eta}}.$$

The normal and tangential derivatives follow the expressions

$$\frac{\partial}{\partial n} = \nabla\cdot\mathbf{n} = \frac{\dfrac{\partial}{\partial\xi}}{\sqrt{\sin^2\eta + \varepsilon^2\cos^2\eta}}, \quad \frac{\partial}{\partial\tau} = \nabla\cdot\boldsymbol{\tau} = \frac{\dfrac{\partial}{\partial\eta}}{\sqrt{\sin^2\eta + \varepsilon^2\cos^2\eta}}.$$

The boundary conditions for potentials at $\xi = \xi_0$ are given as

$$\begin{aligned} &\frac{\partial\varphi}{\partial\xi} + \frac{\partial\psi}{\partial\eta} = \varepsilon u_0 \cos\eta, \\ &\frac{\partial\varphi}{\partial\eta} - \frac{\partial\psi}{\partial\xi} = -u_0 \sin\eta. \end{aligned} \tag{12}$$

The Laplace equation for the potential φ in elliptic coordinates is given by the equality

$$\frac{\partial^2\varphi}{\partial\xi^2} + \frac{\partial^2\varphi}{\partial\eta^2} = 0. \tag{12}$$

Its solution, decreasing at infinity, has the following form:

$$\varphi(\xi,\eta) = \sum_{k=1}^{\infty} C_{2k-1} e^{-(2k-1)(\xi-\xi_0)} \cos(2k-1)\eta. \tag{13}$$

The equation for the potential ψ in elliptic coordinates is expressed by the formula

$$\frac{1}{(\mathrm{ch}^2\xi - \cos^2\eta)}\left(\frac{\partial^2\psi}{\partial\xi^2} + \frac{\partial^2\psi}{\partial\eta^2}\right) + (1-\varepsilon^2)\, i\beta\,\psi = 0. \tag{14}$$

It is known (see monograph [15]) that Eq. (14) allows for separation of variables, and its solution, decreasing at infinity and odd with a period of 2π, has the form

$$\psi(\xi,\eta) = \sum_{k=1}^{\infty} D_{2k-1} H_{2k-1}(\xi,\xi_0,q)\, se_{2k-1}(\eta,q). \tag{15}$$

Here we introduce the notations

$$q = \frac{1}{4} i (1-\varepsilon^2)\beta;$$

$$H_{2k-1}(\xi,\xi_0,q) = \frac{Ne^{(1)}_{2k-1}(\xi,q)}{\left.\dfrac{dNe^{(1)}_{2k-1}(\xi,q)}{d\xi}\right|_{\xi=\xi_0}}; \tag{16}$$

the following functions are used:

$se_{2k-1}(b_{2k-1}(q),\eta,q)$ is an odd Mathieu function with a period of 2π;

$Ne^{(1)}_{2k-1}(b_{2k-1}(q),\xi,q)$ is a modified Mathieu function of the third kind (the terminology adopted in monograph [15] is used).

Both functions correspond to the eigenvalue $b_{2k-1}(q)$, $k = 1, 2, \ldots$.

Note that in addition to (ξ,η), the remaining arguments in all functions are omitted for brevity from now on.

The Mathieu periodic function $se_{2k-1}(\eta)$ and its derivative can be written as a Fourier series:

$$se_{2k-1}(\eta)=\sum_{r=1}^{\infty}B_{2k-1,2r-1}\sin(2r-1)\eta,$$
$$\frac{d}{d\eta}se_{2k-1}(\eta)=\sum_{r=1}^{\infty}(2r-1)B_{2k-1,2r-1}\cos(2r-1)\eta, \quad (17)$$

$$B_{2k-1,2r-1}=\frac{1}{\pi}\int_0^{2\pi}se_{2k-1}(\eta)\sin(2r-1)\eta d\eta. \quad (18)$$

The formulas given in monograph [15] are used for the non-periodic Mathieu function $Ne_{2k-1}^{(1)}(\xi)$:

$$Ne_{2k-1}^{(1)}(\xi)=\frac{s_{2k-1}}{\sqrt{q}}\sum_{r=1}^{\infty}(-1)^{r-1}B_{2k-1,2r-1}\left[J_{r-1}(v_1)H_r^{(1)}(v_2)-J_r(v_1)H_{r-1}^{(1)}(v_2)\right],$$
$$s_{2k-1}=se'_{2k-1}(0)se_{2k-1}\left(\frac{\pi}{2}\right), v_1=\sqrt{q}e^{-\xi}, v_2=\sqrt{q}e^{\xi}. \quad (19)$$

Therefore,

$$H_{2k-1}(\xi,\xi_0)=$$
$$=\frac{\frac{1}{\sqrt{q}}\sum_{s=1}^{\infty}(-1)^{s-1}B_{2k-1,2s-1}\left[J_{s-1}(v_1)H_s^{(1)}(v_2)-J_s(v_1)H_{s-1}^{(1)}(v_2)\right]}{\sum_{s=1}^{\infty}(-1)^{s-1}B_{2k-1,2s-1}\left[J_{s-1}(v_1)H_s^{(1)}(v_2)-J_s(v_1)H_{s-1}^{(1)}(v_2)\right]'\Big|_{\xi=\xi_0}},$$

$$\left[J_{s-1}(v_1)H_s^{(1)}(v_2)-J_s(v_1)H_{s-1}^{(1)}(v_2)\right]'\Big|_{\xi=\xi_0}=$$
$$=e^{-\xi_0}\left[-J'_{s-1}(v_1^0)H_s^{(1)}(v_2^0)+e^{2\xi_0}J_{s-1}(v_1^0)H_s'^{(1)}(v_2^0)+J'_s(v_1^0)H_{s-1}^{(1)}(v_2^0)-e^{2\xi_0}J_s(v_1^0)H_{s-1}'^{(1)}(v_2^0)\right],$$
$$v_1^0=\sqrt{q}e^{-\xi_0}, v_2^0=\sqrt{q}e^{\xi_0}.$$

Substituting solutions (13), (15) into boundary conditions (12), we obtain the following system of equations:

$$\begin{cases}-\sum_{k=1}^{\infty}(2k-1)C_{2k-1}\cos(2k-1)\eta+\sum_{k=1}^{\infty}D_{2k-1}H_{2k-1}(\xi_0,\xi_0)se'_{2k-1}(\eta)=\varepsilon u_0\cos\eta,\\ -\sum_{k=1}^{\infty}(2k-1)C_{2k-1}\sin(2k-1)\eta-\sum_{k=1}^{\infty}D_{2k-1}se_{2k-1}(\eta)=-u_0\sin\eta.\end{cases} \quad (20)$$

If we use the representation $se_{2k-1}(\eta)$, $se'_{2k-1}(\eta)$ as a Fourier series in terms of the functions $\sin(2k-1)\eta$, $\cos(2k-1)\eta$, then we can reduce system of equations (20) with respect to the coefficients C_{2k-1}, D_{2k-1} to an infinite system of linear equations, i.e.,

$$\begin{cases}-C_{2r-1}+\sum_{k=1}^{\infty}D_{2k-1}B_{2r-1,2k-1}H_{2k-1}(\xi_0,\xi_0,q)=\varepsilon u_0\delta_r^1,\\ C_{2r-1}+\sum_{k=1}^{\infty}D_{2k-1}B_{2r-1,2k-1}=u_0\delta_r^1,\end{cases}$$

where δ_r^1 is the Kronecker symbol, denoting the following:

$$\delta_r^1 = \begin{cases} 1, & r = 1, \\ 0, & r \neq 1. \end{cases}$$

Finally, excluding the coefficients C_{2r-1}, we obtain

$$\sum_{k=1}^{\infty} D_{2k-1} B_{2r-1,2k-1} \left(1 + H_{2k-1}\left(\xi_0, \xi_0, q\right)\right) = \left(1+\varepsilon\right) u_0 \delta_r^1. \tag{21}$$

The infinite system of linear equations (21) can be solved by the reduction method.

Finally, in accordance with Eq. (6), we have:

$$\begin{gathered} F_x = -i \int_0^{2\pi} \sum_{k=1}^{\infty} C_{2k-1} \cos\left(2k-1\right)\eta \cos\eta d\eta - \\ -i \int_0^{2\pi} \sum_{k=1}^{\infty} D_{2k-1} H_{2k-1}\left(\xi_0, \xi_0, q\right) se_{2k-1}\left(\eta, q\right) \sin\eta d\eta = \\ = i\pi\left(-C_1 - \sum_{k=1}^{\infty} D_{2k-1} H_{2k-1}\left(\xi_0, \xi_0, q\right) B_{2k-1,1}\right). \end{gathered} \tag{22}$$

The asymptotic form of the solution for large β. As shown in [16], the method of boundary integral equations for $\beta \to \infty$ can be used to construct an asymptotic solution of the Stokes problem for an arbitrary smooth convex contour as asymptotic series in powers of $1/\sqrt{\beta}$.

The first two terms of such an expansion give expressions for potentials at contour points [16]:

$$\begin{gathered} \varphi\left(\xi_0, \eta\right) \underset{\beta\to\infty}{\sim} -u_0 \varepsilon \cos\eta + \frac{1}{i\sqrt{i\beta}} \frac{u_0\left(1+\varepsilon\right)\varepsilon^2}{\pi} \sum_{m=1}^{\infty} \frac{\mathrm{I}_m\left(\varepsilon\right)}{m} \cos m\eta, \\ \mathrm{I}_m\left(\varepsilon\right) = \int_0^{2\pi} \frac{\cos\eta \cos m\eta}{\left(\varepsilon^2 \cos^2\eta + \sin^2\eta\right)^{3/2}} d\eta, \\ \psi\left(\xi_0, \eta\right) \underset{\beta\to\infty}{\sim} \frac{1}{i\sqrt{i\beta}} \frac{\left(1+\varepsilon\right) u_0 \sin\eta}{\sqrt{\varepsilon^2 \cos^2\eta + \sin^2\eta}}. \end{gathered} \tag{23}$$

Substituting Eqs. (23) into expression (6) for the hydrodynamic force produces the following formula:

$$F_x = i\pi u_0 \varepsilon + \frac{8u_0\varepsilon}{\sqrt{i\beta}\left(1-\varepsilon\right)} \left(K\left(\frac{\varepsilon^2-1}{\varepsilon^2}\right) - E\left(\frac{\varepsilon^2-1}{\varepsilon^2}\right)\right), \tag{24}$$

where K, E are complete elliptic integrals.

At $\varepsilon \to 1$, the ellipse degenerates into a circle and Eq. (24) becomes Eq. (9).

Asymptotic form of the solution for small β.If $\beta \to 0$**, we** have:

$$q \to 0,\ se_{2k-1}\left(\eta, q\right) \to \sin\left(2k-1\right)\eta,\ B_{2k-1,2r-1} \underset{q\to 0}{\sim} \delta_{2r-1}^{2k-1}. \quad [15]$$

and then the first term in Eqs. (13), (15) becomes the principal one. Therefore, system of equations (20) allows to obtain the following dependences:

$$\varphi\left(\xi, \eta\right) \underset{q\to 0}{\sim} \frac{H_1\left(\xi_0, \xi_0, q\right) - \varepsilon}{H_1\left(\xi_0, \xi_0, q\right) + 1} e^{-\left(\xi-\xi_0\right)} u_0 \cos\eta,$$

$$\psi(\xi,\eta) \underset{q\to 0}{\sim} \frac{(1+\varepsilon)H_1(\xi,\xi_0,q)}{H_1(\xi_0,\xi_0,q)+1} u_0 \sin\eta.$$

Introducing an expansion in series with respect to a small parameter β, we find:

$$\varphi \underset{\beta\to 0}{\sim} \frac{8ie^{-\xi-\xi_0}}{(1-\varepsilon)\beta\left\{-e^{-4\xi_0}-e^{-2\xi_0}+\left[2\gamma-i\pi+\ln\beta+2\ln\left(e^{\xi_0}\frac{1}{4}\sqrt{i(1-\varepsilon^2)}\right)\right]\right\}} u_0\cos\eta,$$

$$\psi \underset{\beta\to 0}{\sim} \frac{8ie^{-\xi-\xi_0}}{(1-\varepsilon)\beta\left\{-e^{-4\xi_0}-e^{-2\xi_0}+\left[2\gamma-i\pi+\ln\beta+2\ln\left(e^{\xi_0}\frac{1}{4}\sqrt{i(1-\varepsilon^2)}\right)\right]\right\}} u_0\sin\eta,$$

consequently, the force is expressed as follows in accordance with Eq. (6):

$$F_x \underset{\beta\to 0}{\sim} \frac{16\pi u_0 e^{-2\xi_0}}{(1-\varepsilon)\beta\left\{-e^{-4\xi_0}-e^{-2\xi_0}+\left[2\gamma-i\pi+\ln\beta+2\ln\left(e^{\xi_0}\frac{1}{4}\sqrt{i(1-\varepsilon^2)}\right)\right]\right\}}. \tag{25}$$

If ε → 1,

$$2\ln\left(e^{\xi_0}\frac{1}{4}\sqrt{i(1-\varepsilon^2)}\right) = 2\ln\left(\frac{1+\varepsilon}{4}\sqrt{i}\right) = \frac{i\pi}{2} - 2\ln 2,$$

$$e^{-2\xi_0} \sim \frac{1-\varepsilon}{2} + O(1-\varepsilon)^2,$$

Eq. (11) for the circle is obtained from expression (25) by passage to the limit.

Finite difference method. The numerical algorithm for solving the problem is described in detail in [17]. Comparing the results obtained by different methods in [16, 17], we can conclude that the proposed finite difference method can be applied in a fairly wide range of values β ∈[0.1,25], i.e., even in the range of 'moderate' values β ∈ [1,10], where the applicability of asymptotic solutions is questionable.

Numerical implementation of solution methods

To numerically implement the formulas of the analytical solution, it is necessary to calculate the eigenvalues $b_{2k-1}(q)$ of the Mathieu equation and the coefficients $B_{2k-1,2r-1}$. The technique for describing them has long been described [15, 18]. Although the general consensus is that theoretical work on the methods for calculating these eigenvalues is completed [19], in reality, many results have not yet been obtained, especially for the case of purely imaginary and modulo large values of *q*. Therefore, it is necessary to rely on the Mathematica package, capable of calculating periodic Mathieu functions, and to find the coefficients $B_{2k-1.2r-1}$ by numerical integration by Eqs. (18).

The experience gained from calculations by the methods described above indicates that the range of eigenvalues β can be conditionally divided into three parts: small (β < 1), large (β > 10) and moderate (β ∈ [1,10]). The reduction method for system (21) converges quickly in the range of small β (in 3–5 iterations). Fig. 1 shows the calculated magnitudes of the force $|F_x|$ for different ε.

The analysis showed that convergence of the method slows down at β > 1, the dimension of the matrix of coefficients $B_{2k-1.2r-1}$ increases, the condition number increases rapidly, and it becomes practically impossible to calculate the coefficients $B_{2k-1.2r-1}$ by Eqs. (18) at β > 10, since the integrand oscillates rapidly. The problems with implementing the calculation scheme in this range are generally similar to those in the theory of diffraction and scattering of high-frequency sound waves.

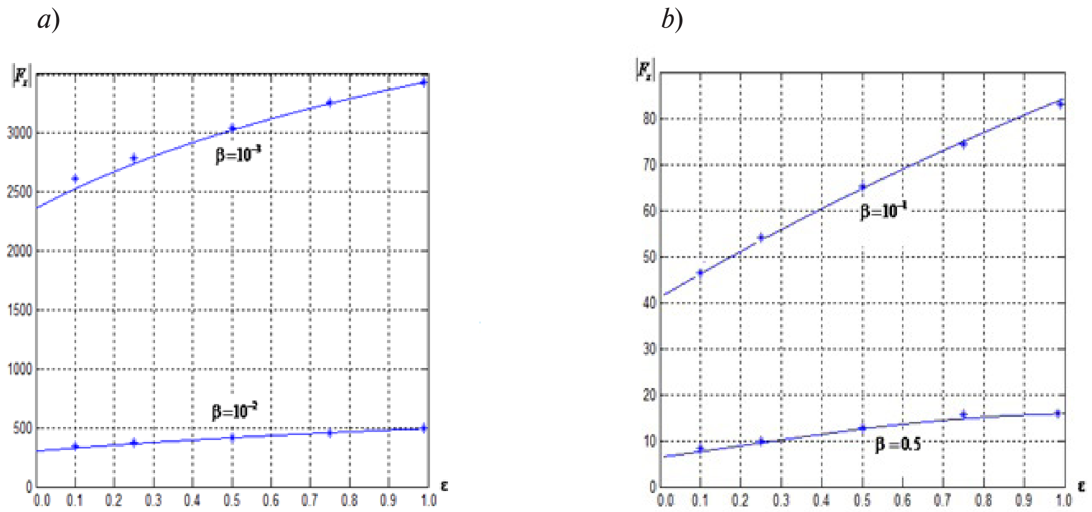

Fig. 1. Calculated dependences of the force magnitudes $|F_x|$ on the parameter ε for different small β: 10^{-3}, 10^{-2} (*a*) and 10^{-1}, 0.5 (*b*).
Eq. (25) (solid lines) and analytical solution (symbols) are used

In the case of the Stokes problem for a circular cylinder, combining the asymptotes at $\beta < 1$ and $\beta > 10$ with the finite difference method allows to cover the entire possible variation range of the parameter β.

The same effect is likely to be expected for the elliptical contour. It was found in [16] by the finite difference method that the asymptote (24) gives acceptable results already at $\beta > 10$.

Fig. 2 shows the calculated results obtained by Eq. (24) and the analytical solution for $\beta \in [1,10]$. It is clear that the solution arrives at the given asymptote.

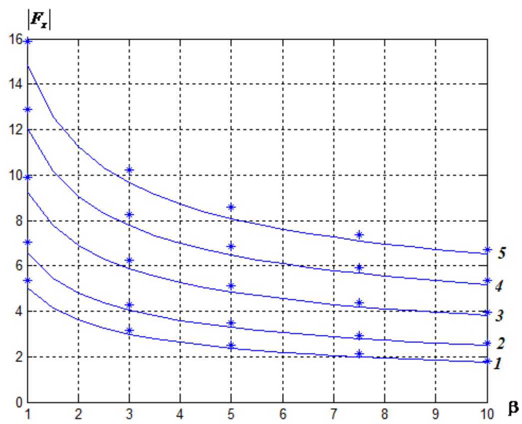

Fig. 2. Calculated dependences of force magnitude $|F_x|$ on the parameter β for different values of the parameter ε: 0.10 (*1*), 0.25 (*2*), 0.50 (*3*), 0.75 (*4*), 0.99 (*5*).
Eq. (24) (solid lines) and analytical solution (symbols) were used

Conclusion

The methods considered in this paper can be used to solve the problem of small harmonic oscillations of an elliptical contour immersed in incompressible viscous liquid. Comparing the results, we can conclude they can be used in combination to describe the solution in almost the entire range of the β parameter values.

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