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A NUMERICAL ALGORITHM FOR SYNTHESIZING THE POLYNOMIALS DEVIATING LEAST FROM ZERO WITH A GIVEN WEIGHT

A. S. Berdnikov[1]✉, *K. V. Solovyev*[2, 1]

[1] Institute for Analytical Instrumentation RAS, St. Petersburg, Russia;

[2] Peter the Great St. Petersburg Polytechnic University, St. Petersburg, Russia

✉ asberd@yandex.ru

Abstract. The article considers numerical algorithms for determining the coefficients of polynomials with a fixed leading coefficient, the algorithms supplying a minimum deviation from zero in a minimax norm with a given weight function. The polynomials serve as a useful tool in many numerical methods, in particular, in the Lanczos' tau method which provides an approximate numerical analytic solution of ordinary differential equations with coefficients as polynomials in the independent variable. The well-known Chebyshev polynomials determined analytically are the special case of such polynomials, however, in most cases of weight functions, such polynomials can only be determined and tabulated numerically.

Keywords: minimax norm, Chebyshev polynomial, optimal approximation, interpolation, numerical algorithm

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ЧИСЛЕННЫЙ АЛГОРИТМ ДЛЯ КОНСТРУИРОВАНИЯ МНОГОЧЛЕНОВ, НАИМЕНЕЕ ОТКЛОНЯЮЩИХСЯ ОТ НУЛЯ С ЗАДАННЫМ ВЕСОМ

А. С. Бердников[1]✉, *К. В. Соловьев*[2, 1]

[1] Институт аналитического приборостроения РАН, Санкт-Петербург, Россия;

[2] Санкт-Петербургский политехнический университет Петра Великого,

Санкт-Петербург, Россия

✉ asberd@yandex.ru

Аннотация. В статье рассматриваются численные алгоритмы для определения коэффициентов многочленов с фиксированным старшим коэффициентом, которые обеспечивают на заданном интервале минимальное отклонение от нуля в минимаксной норме с заданной весовой функцией. Указанные многочлены служат полезным инструментом во многих численных методах, в частности в тау-методе Ланцоша, обеспечивающего нахождение приближенного численно-аналитического решения обыкновенных дифференциальных уравнений с коэффициентами в виде многочленов от независимой переменной. Частным случаем таких многочленов являются хорошо известные многочлены Чебышева, определяемые аналитически, однако в большинстве случаев весовых функций такие многочлены можно определить и табулировать только численно.

Ключевые слова: минимаксная норма, многочлен Чебышева, оптимальная аппроксимация, интерполяция, численный алгоритм

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Introduction

Approximations of functions that are optimal by the minimax (uniform) norm offer significant advantages over the simpler approximation of functions by the least squares method [1]. Expansion of a function into a truncated series consisting of polynomials of least deviation from zero is commonly used to construct such approximations [2, 3, 5–7]. In addition, the polynomials of least deviation from zero are useful for making an optimal choice of collocation points for interpolation of functions by polynomials (these points are zeros of the corresponding polynomials [1]), as well as for constructing approximate solutions for linear ordinary differential equations (ODE) with coefficients in the form of polynomials with respect to an independent variable [1, 2, 4].

Unfortunately, there are not many polynomials of least deviation from zero at a given interval, for which there is an explicit algebraic representation in analytical form. In this paper, we consider a refined version of a rapidly convergent numerical algorithm for calculating the coefficients of polynomials of least deviation from zero over a given interval with a given weight, which was partially discussed in [8].

Polynomials of least deviation from zero

The problem on constructing a polynomial of a given degree that deviates least from zero in a given interval with a given weight is formulated as follows. Let there be a continuous function $f(x)$ and a finite interval $[a, b]$ for which a continuous weight function $q(x)$ is strictly positive in this interval; however, the ends of the interval where the function $q(x)$ can vanish[1] may be the exceptions. It is required to find a polynomial of degree n of the form

$$p(x) = a_0 + a_1x + a_2x^2 + \ldots + a_{n-1}x^{n-1} + a_nx^n \quad (1)$$

with previously undetermined coefficients a_0, a_1, a_2, ..., a_{n-1} and the leading coefficient $a_n = 1$, which is the solution to the optimization problem

$$\max |q(x)p(x)| \to \min. \quad (2)$$

Here, maximization is performed with respect to the variable $x \in [a, b]$, and minimization with respect to the coefficients a_0, a_1, a_2, ..., a_{n-1}. Such a polynomial is called the polynomial of least deviation from zero in the interval $[a, b]$ with the weight $q(x)$.

First-kind Chebyshev polynomials taking the following form provide the smallest deviation from zero in the interval $[-1, +1]$ with the weight $q(x) = 1$:

$$T_n(x) = \cos[n \arccos(x)];$$

the polynomials are scaled using a 2^{-n} multiplier so that the leading coefficient of the polynomial is equal to unity. Scaled second-kind Chebyshev polynomials of the second kind taking the following form provide the smallest deviation from zero in the interval $[-1, +1]$ with the weight $q(x) = (1 - x^2)^{1/2}$:

$$U_n(x) = \sin[(n + 1) \arccos(x)] / (1 - x^2)^{1/2}.$$

Polynomials of degree n, providing the smallest deviation from zero in the interval [0, 1] with the weight $q(x) = x$, are obtained by separating a multiplier in the form of a weight function from first-kind Chebyshev polynomials of degree $n + 1$, for which such an argument substitution $x \to ax + b$ should be performed that the interval $[x_a + 1]$ of argument values be mapped to the interval [0, 1], where

$$x_a = \cos[\pi(2n + 1)/(2n + 2)]$$

is the minimum zero of the function $T_{n+1}(x)$.

Polynomials of degree n, providing the smallest deviation from zero in the interval [0, 1] with the weight $q(x) = x(1 - x)$, are obtained by separating a multiplier in the form of a weight function from first-kind Chebyshev polynomials oof degree $n + 2$, for which such an argument substitution $x \to ax + b$ should be performed that the interval $[x_b, x_c]$ of argument values is mapped to the interval [0, 1], where

$$x_b = \cos[\pi(2n + 3) / (2n + 4)] \text{ and } x_c = \cos[\pi/(2n+4)]$$

are the minimum and maximum zeros of the function $T_{n+2}(x)$.

Other examples of polynomials of least deviation from zero are given in monographs [6, 7], but in general, very few such polynomials for which there are explicit algebraic expressions are known.

Chebyshev criterion

The fundamental properties of polynomials of least deviation from zero are determined by the following statement.

[1] A weight function with zeros in the interval $[a, b]$ requires that the Chebyshev maxima and minima (and the test points for the numerical algorithm) do not coincide with the zeros of the weight function, and the signs of alternating maxima and minima change in accordance with the sign of the weight function. Furthermore, the interval $[a, b]$ can be infinite on the right and/or on the left-hand sides, but the weight function $q(x)$ should tend to zero at infinity no slower than the power function $1/x^k$ [5–7].

Statement 1 (Chebyshev criterion for polynomials of least deviation from zero). *In order for a polynomial p(x) of degree n with the leading coefficient equal to unity to be a solution to problem (2), it is necessary and sufficient that there exist such a set of n + 1 points* $x_0 < x_1 < x_2 < \ldots < x_n$ *belonging to the interval [a,b], and such a number* ε*(positive or negative) that the following conditions are satisfied:*

$$-|\varepsilon| \leq q(x)p(x) \leq |\varepsilon| \text{ for } x \in [a, b], \tag{3}$$

$$q(x_k)p(x_k) = (-1)^k\varepsilon \text{ for } k = 0, 1, 2, \ldots, n. \tag{4}$$

This criterion is a special case of a more general statement about uniform approximations by rational functions, considered by Pafnuty Chebyshev in his mémoire [17] (see also [6, 7]). The mémoire was not published, so it is difficult to establish the exact date when this result was obtained. Notably, Chebyshev also turned to the problem of the best uniform approximation by polynomials in his earlier works (see, for example, [16]), frequently revisiting the statements formulated in [17] later (see, for example, [18]).

Below we present the modern proof of the Chebyshev criterion for polynomials deviating least from zero, divided for convenience of presentation into several auxiliary statements 2–6: lower and upper bound estimate of norm (2), existence of optimal solution, sufficiency of Chebyshev criterion, necessity of Chebyshev criterion, uniqueness of solution.

Evidently, provided that conditions (3), (4) are satisfied, the equality $\max |q(x)p(x)| = |\varepsilon|$ holds true. The points $x_0 < x_1 < x_2 < \ldots < x_{n+1}$ of the interval $[a, b]$ with alternating positive minima and negative maxima equal in absolute value to the maximum absolute value of the function considered are called the Chebyshev alternation. According to Chebyshev's alternation theorem ([6, 7] and [16–20]), the condition under consideration is necessary and sufficient for the polynomial $p(x)$ to be a solution to optimization problem (2), so that such a solution always exists and is unique.

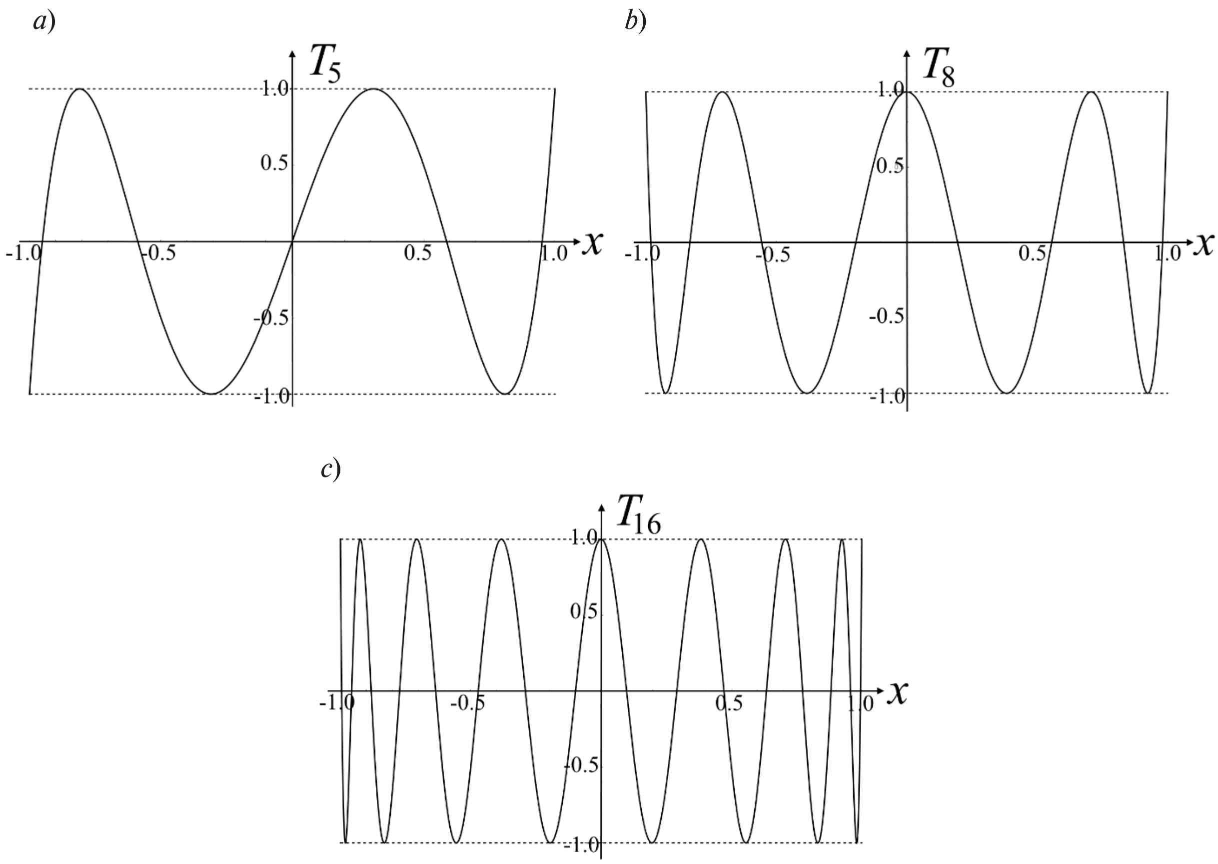

Fig. 1. Chebyshev polynomials $T_n(x)$ for $n = 5$ (*a*), 8 (*b*) and 16 (*c*), deviating least from zero in the segment [−1, +1]

Fig. 1 shows graphs of first-kind Chebyshev polynomials [1–3], illustrating that this condition is satisfied in the interval [–1, +1] with weight $q(x) = 1$.

In accordance with conditions (3), (4) for an optimal polynomial $p(x)$ with a coefficient at the highest degree equal to unity, there is a set of $n + 1$ test points x_k in which the deviation from zero $q(x)p(x)$ has alternating local negative minima and positive maxima, equal to $\pm|\varepsilon|$, and the values of these minima and maxima are global in the interval $[a, b]$.

This criterion is a special case of Chebyshev's theory of minimax approximation using rational functions [6, 7], however, the proof of the corresponding statements is simplified for polynomials of least deviation from zero.

Statement 2 (de la Vallée Poussin theorem [6, 7, 21–23]). *If the function q(x)p(x) takes values* $\lambda_0, \lambda_1, \ldots, \lambda_{N-1}$ *different than zero with alternating signs, where* $N \geq n + 1$, *at N consecutive points* $x_0 < x_1 < \ldots < x_{N-1}$ *of the interval [a,b] for some polynomial p(x) of degree n with the leading coefficient equal to unity, then the following relation holds true for any other polynomial r(x) of degree n with the leading coefficient equal to unity:*

$\max |q(x)r(x)| \geq \min \{|\lambda_0|, |\lambda_1|, \ldots, |\lambda_{N-1}|\}$.

P r o o f. Suppose there is a polynomial $r(x)$ for which the condition $\max |q(x)r(x)| < \min (|\lambda_0|, |\lambda_1|, \ldots, |\lambda_{N-1}|)$ is satisfied. We assume that the values of $q(x_k)p(x_k)$ in the sequence x_k are strictly positive for even points, and strictly negative for odd points. (The reasoning is similar when the values are positive for odd points and negative for even points). This means that the following condition is satisfied at even points x_k:

$$q(x_k)r(x_k) < |\lambda_k| = q(x_k)p(x_k),$$

and the following condition is satisfied at odd points x_k:

$$q(x_k)r(x_k) > -|\lambda_k| = q(x_k)p(x_k).$$

As a result, the quantity $q(x)[p(x) - r(x)]$ is strictly positive at even points x_k and strictly negative at odd points x_k. The values of $q(x_k)$ do not vanish and, therefore, are strictly positive. The polynomial $p(x) - r(x)$ of degree $n - 1$, which is not identically zero, alternately takes positive and negative values for at least $n + 1$ points and, therefore, has at least n zeros. Consequently, the polynomial $r(x)$ for which

$$\max |q(x)r(x)| < \min (|\lambda_0|, |\lambda_1|, \ldots, |\lambda_{N-1}|),$$

does not exist.

Statement 2 is proved.

Remark. De la Vallée Poussin theorem allows to obtain a lower-bound estimate for solving optimization problem (2). If $\lambda_0, \lambda_1, \ldots, \lambda_{N-1}$ are local maxima and minima of the function $q(x)$ $r(x)$ with alternating signs, then such a structure is called the de la Vallée Poussin alternation; it gives not only the lower-bound estimate for the solution to optimization problem (2), but also the upper-bound estimate, equal to $\max \{|\lambda_0|, |\lambda_1|, \ldots, |\lambda_{N-1}|\}$.

Statement 3. *There is a polynomial p(x) that provides a solution to optimization problem (2).*

This statement is important because it excludes the case when there are polynomials $p_k(x)$ with progressively decreasing values $P_k = \max |q(x)p_k(x)|$, while the minimum of this quantity is never reached in the set of polynomials of fixed degree. In particular, there is no solution to optimization problem (2) without imposing an a priori restriction on the degree of the polynomial $p(x)$.

P r o o f. The values of the quantities $P_k = \max |q(x)p_k(x)|$ are bounded from below by zero, so there is an exact lower bound $P$for the values of P_k in the set of polynomials. According to the definition of the exact lower bound, there is a sequence of polynomials $p_k(x)$ of degree n:

$$p_k(x) = a_{0,k} + a_{1,k}x + a_{2,k}x^2 + \ldots + a_{n-1,k}x^{n-1} + x^n,$$

for which $P \leq P_k \leq 2P$ and $\lim P_k = P$ with $k \to \infty$.

Consequently, the values of the polynomials $p_k(x)$ in the interval $[a + \delta, b - \delta]$ (where δ is sufficiently small) starting from some number k are bounded from above and from below (an offset had to be made from the ends of the interval to take into account the case when the continuous weight function $q(x)$ at the ends of the interval can equal zero). It follows from the Lagrange formula [15] for a polynomial $p(x)$ of degree n, taking the values y_k at the given $n + 1$ points x_k, of the form

$$p(x) = \sum_{j=1,n+1} \left(y_j \prod_{i=1,n+1,i \neq j} \frac{(x - x_i)}{(x_j - x_i)} \right),$$

that when the values of the polynomial y_k are bounded from above and from below at $n + 1$ fixed points x_k, each individual coefficient of the polynomial is also bounded from above and from below. According to the Bolzano–Weierstrass theorem (lemma) on the limit point, each infinite bounded sequence of points in the space R^n has an infinite subsequence with a limit. Therefore, a subsequence with a limit for each of the polynomial coefficients can be selected in the considered sequence of polynomials $p_k(x)$ (represented as vectors of length $n + 1$, consisting of the polynomial coefficients bounded from above and from below).

In other words, there is a sequence of polynomials $p_k(x)$ for which lim $P_k = P$ for $k \to \infty$, and the coefficients of the polynomials $p_k(x)$ have limit values $b_j = \lim a_{j,k}$ при $k \to \infty$. Evidently, this condition also holds true for the higher coefficients of the polynomials $p_k(x)$, by definition equal to unity.

Consider a polynomial $r(x)$ with the coefficients b_j:

$$r(x) = b_0 + b_1x + b_2x^2 + \ldots + b_{n-1}x^{n-1} + x^n.$$

Since $b_j = \lim a_{jk}$, the following condition is satisfied for any fixed value of x:

$$\lim p_k(x) = r(x) \text{ for } k \to \infty,$$

so that convergence to the limit is uniform over the considered finite interval of values of x. It follows from the inequality

$$\max |q(x)r(x)| = \max |q(x)\{p_k(x) + [r(x) - p_k(x)]\}| \leq$$

$$\leq \max |q(x)p_k(x)| + \max |q(x)| \cdot \max |r(x) - p_k(x)|$$

that $\max |q(x)r(x)| = P$. Indeed, $\max |q(x)r(x)|$ cannot be less than P; at the same time, the first term on the right-hand side of the inequality tends to P for $k \to \infty$, and the second term tends to zero. Thus, quantity (2) reaches its lower bound P for the polynomial $r(x)$.

Statement 3 is proved.

Statement 4. *If conditions (3), (4) of the Chebyshev criterion are satisfied for the polynomial p(x), then the optimal value for the right-hand side of optimization problem (2) is equal to |ε|, and the polynomial p(x) (possibly one of many) ensures that this optimum is achieved.*

Proof. The condition $\max|q(x)p(x)| = |\varepsilon|$ is satisfied for a polynomial $p(x)$ satisfying the Chebyshev criterion, so that the solution of optimization problem (2) does not exceed $|\varepsilon|$. However, since the Chebyshev alternation is a special case of the de la Vallée Poussin alternation, then, according to the de la Vallée Poussin theorem (see Statement 2), the solution to optimization problem (2) cannot be less than $|\varepsilon|$. Therefore, the solution of optimization problem (2) is equal to $|\varepsilon|$, and the polynomial $p(x)$ ensures that this optimum is achieved.

Statement 4 is proved.

Statement 5. *The polynomial p(x) providing a solution to optimization problem (2) must satisfy the Chebyshev criterion (see Statement 1).*

Proof. Consider the behavior of the function $F(x) = q(x)p(x)$ in the segment $[a, b]$. The function $q(x)$ can vanish only at the ends of the interval, so the number of zeros of the function $F(x)$ within the interval certainly does not exceed n, and all zeros are isolated points.

Let $y_1, y_2, \ldots, y_m$ be an ordered set of zeros of odd multiplicity for this continuous function (that might not contain a single point). The points $y_1, y_2, \ldots, y_m$ divide the segment $[a, b]$ into $m + 1$ intervals, so that the function $F(x)$ takes alternately a positive or a negative value in each of them. If the function $F(x)$ has no zeros, then the entire segment $[a, b]$ is an interval where the function $F(x)$ is either positive or negative.

For intervals with positive values of $F(x)$, we select a point with the maximum value of the function in this interval (it might not be the only one in this interval), and for intervals with negative values of $F(x)$, we select a point with the minimum value of the function in this interval. We obtain a set of points $y_1, y_2, \ldots, y_m$, dividing the segment $[a, b]$ into $m + 1$ intervals, where the function $F(x)$ preserves its sign, but changes it upon intersecting the interval boundary. The set of points $x_0, x_1, x_2, ..., x_m, x_{m+1}$, belonging to these intervals, which do not coincide with the ends of the intervals, consists of alternating positive maxima and negative minima $\lambda_0, \lambda_1, \lambda_2, ..., \lambda_{m+1}$ of the function $F(x)$.

Let $\lambda = \max|\lambda_k|$. If the maximum is equal to λ in any interval, and the minimum modulus is less than λ in the adjacent (subsequent or preceding) interval, then this interval is attached to the current interval together with the subsequent interval with a positive maximum, regardless of its value (Fig. 2,*a*). The resulting interval has the following property: such a positive constant can be subtracted from the function $F(x)$ that the positive maximum of the function in the interval decreases, but the negative minimum modulus of the function in the interval does not increase sufficiently to exceed the new positive maximum.

Similarly, if the minimum is equal to $-\lambda$ in some interval, and the maximum is less than λ in the neighboring interval, then this first interval is attached the current interval together with the subsequent interval containing a negative minimum (see Fig. 2,*b*). A positive constant can be added to the function $F(x)$ for this interval, which would reduce the negative minimum modulus of the function in the interval and simultaneously increase the positive maximum of the function in the interval so as not to exceed the new negative minimum modulus.

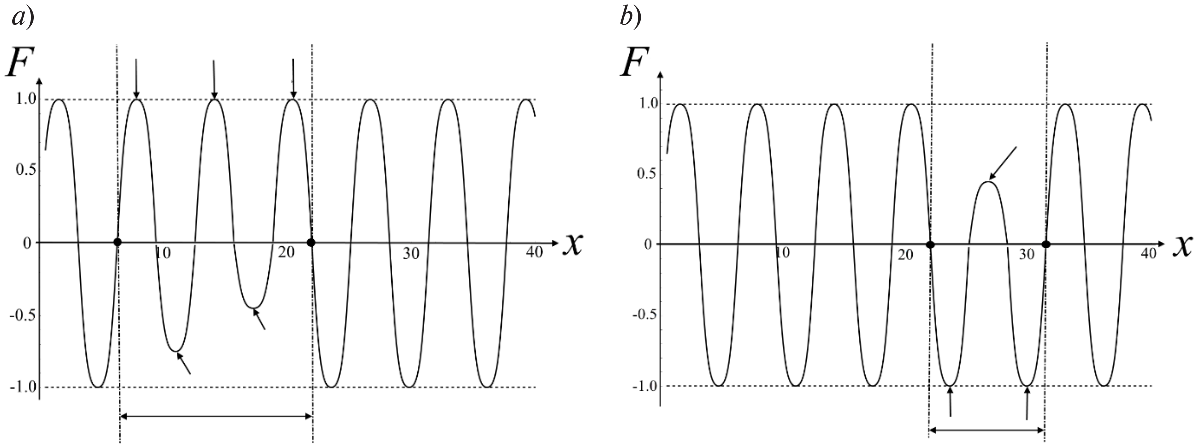

Fig. 2. Procedure for combining segments with alternating maxima and minima to construct the Chebyshev alternation (the length of the combined segments is marked by the extension lines): single or several sequentially located minima lie above the minimum level of the function $F(x)$ (*a*); single or several sequentially located maxima lie below its maximum level (*b*). The extrema are shown by arrows

Ultimately, a set of N intervals is obtained in the interval $[a, b]$ for a given polynomial $p(x)$, containing points $x_0, x_1, x_2, ..., x_{N-1}$ with alternating positive maxima and negative minima of the function $F(x)$ equal to $\pm\varepsilon$, where $\varepsilon = \max |q(x)p(x)|$. On the other hand, if N is greater than or equal to $n + 1$, then such a polynomial $p(x)$ satisfies the Chebyshev criterion and, in accordance with statement 2, is the solution to optimization problem (2).

Now let N be less than or equal to n. At the stage when the intervals were combined, the boundaries of the new intervals were determined, i.e., a set of points $y_1, y_2, ..., y_{N-1}$, where the function $F(x)$ vanishes and changes its sign. There are also constants $\delta_k > 0$ and a universal constant $\delta = \min \delta_k$. These constants can be subtracted from the function $F(x)$ in intervals with positive maxima or added to the function $F(x)$ in intervals with negative minima, reducing the minimax norm in the corresponding interval.

Let the function $F(x)$ take positive values in the first interval (the reasoning is similar if the function $F(x)$ takes negative values in the first interval). Consider the function $Q(x)$, which is the product of a polynomial of degree no higher than $n - 1$ and the weight function $q(x)$:

$$Q(x) = q(x)(y_1 - x)(y_2 - x) \cdots (y_{N-1} - x).$$

The function $Q(x)$ is strictly positive in the intervals where there is a Chebyshev maximum of the function $F(x)$, and strictly negative in those where there is a Chebyshev minimum of this function.

Let $R = \max |Q(x)|$. If we subtract the function $sQ(x)$ with a sufficiently small positive factor s from the function $F(x)$ (for example, we can choose $s = (\delta/R)$), then this should safely reduce the positive maxima of the function $F(x)$ and reduce its negative minima, thus also serving to reduce $\max |F(x)|$. Therefore, the considered polynomial $p(x)$ cannot serve as a solution to optimization problem (2), since an even smaller value for $\max |F(x)|$ can be obtained to replace the current polynomial $p(x)$ of degree n (its leading coefficient is unity) with a new polynomial $r(x)$ of degree n, where the leading coefficient is also equal to unity:

$$G(x) = F(x) - (\delta/R)Q(x) = q(x)p(x) - (\delta/R)q(x)(y_1 - x) \cdots (y_{N-1} - x) =$$

$$= q(x)[p(x) - (\delta/R)(y_1 - x) \cdots (y_{N-1} - x)] = q(x)r(x),$$

$$\max |G(x)| < \max |F(x)|.$$

Thus, the optimal polynomial of degree n, considered in statement 3, must satisfy the Chebyshev criterion.

Statement 5 is proved.

Statement 6. *The polynomial p(x) satisfying the Chebyshev criterion and providing a solution to optimization problem (2) is unique.*

Proof. Let there be two polynomials $p(x)$ and $r(x)$ of degree n, whose higher coefficients are equal to unity and which are the solution to optimization problem (2). Since the polynomials $p(x)$ and $r(x)$ are not identically zero, the expressions $q(x)p(x)$ and $q(x)r(x)$ have non-zero maxima and minima in the interval $[a, b]$.

According to statement 5, each of the polynomials $p(x)$ and $r(x)$ satisfies the Chebyshev criterion, and the value $\varepsilon \neq 0$ is the same for them. Polynomials of the form

$$s(x,\alpha) = (1 - \alpha)p(x) + \alpha r(x)$$

are polynomials of degree n with a unit coefficient for the highest degree. These polynomials are also solutions to optimization problem (2) for $0 < \alpha < 1$: the chain of inequalities

$$|q(x)s(x,\alpha)| = |q(x)[(1 - \alpha)p(x) + \alpha r(x)]| \leq (1 - \alpha)|q(x)p(x)| + \alpha|q(x)r(x))| \leq |\varepsilon|$$

implies that $\max |q(x)s(x,\alpha)| \leq |\varepsilon|$, and since the case $\max |q(x)s(x,\alpha)| < |\varepsilon|$ is impossible because $|\varepsilon|$ is the solution to optimization problem (2), we obtain that $\max |q(x)s(x,\alpha)| = |\varepsilon|$.

Let $x_1, x_2, \ldots, x_{n+1}$ be Chebyshev inflection points with alternating maxima and minima of the function $q(x)s(x,\alpha)$, equal to $\pm\varepsilon$. If x_k is the point of the negative minimum of the function $q(x)s(x,\alpha)$, equal to $-|\varepsilon|$, then the conditions

$$q(x_k)s(x_k,\alpha) = (1 - \alpha)q(x_k)p(x_k) + \alpha q(x_k)r(x_k)) = -|\varepsilon|,$$

$$q(x_k)p(x_k) \geq -|\varepsilon|,\ q(x_k)r(x_k) \geq -|\varepsilon|,\ 0 < \alpha < 1$$

imply that the case when

$$q(x_k)p(x_k) = -|\varepsilon| \text{ and } q(x_k)r(x_k) = -|\varepsilon|,$$

is the only possible one.

Similarly, if x_k is the point of the positive maximum of the function $q(x)s(x,\alpha)$, equal to $|\varepsilon|$, then the conditions

$$q(x_k)s(x_k,\alpha) = (1 - \alpha)q(x_k)p(x_k) + \alpha q(x_k)r(x_k)) = |\varepsilon|,$$

$$q(x_k)p(x_k) \leq |\varepsilon|,\ q(x_k)r(x_k) \leq |\varepsilon|,\ 0 < \alpha < 1,$$

imply that

$$q(x_k)p(x_k) = |\varepsilon| \text{ and } q(x_k)r(x_k) = |\varepsilon|.$$

Since the values of the polynomials $p(x)$ and $r(x)$ of degree n with the leading coefficient equal to unity are the same at $n + 1$ different points $x_1, x_2, ..., x_{n+1}$, these polynomials are identically equal to each other.

Algorithm for constructing a polynomial of least deviation from zero

An algorithm for numerically constructing polynomials of least deviation from zero, which is a modified and optimized Remez algorithm [9–14], follows from the above Chebyshev criterion (Statement 1).

Step 1. The initial set of points $x_0 < x_1 < x_2 < ... < x_n$ belonging to the interval $[a, b]$ is chosen arbitrarily.

Step 2. For the current set of points,

$$a \le x_0 < x_1 < x_2 < \ldots < x_n \le b$$

a polynomial $p(x)$ of degree n is constructed for which the conditions

$$q(x_k)p(x_k) = (-1)^k \text{ for } k = 0, 1, 2, ..., n$$

are satisfied.

Evidently, such a polynomial exists, is defined uniquely and can be calculated explicitly by the Lagrange formula [15].

Step 3. The roots belonging to the interval $[a, b]$ are found for the polynomial $p(x)$. Since the values of the polynomial $p(x)$ have different signs at the ends of the intervals $[x_k, x_{k+1}]$, there is at least one root of the polynomial $p(x)$ inside each such interval. There are exactly n such intervals, so there is exactly one root in each of the intervals $[x_k, x_{k+1}]$, and all the roots of the polynomial $p(x)$ are real, belong to the interval $[a, b]$ and are not multiples. It is not difficult to find these roots using a suitable numerical method.

Step 4. The list of roots

$$a < y_1 < y_2 < \ldots < y_n < b,$$

belonging to the interval $[a, b]$, which were found in Step 3, are supplemented by the beginning and end of the interval $y_0 = a$ and $y_{n+1} = b$. Since the polynomial $p(x)$ of degree n preserves the sign in the intervals $[y_k, y_{k+1}]$, either a global maximum (for positive values of the polynomial) or a global minimum (for negative values of the polynomial) exists in each of the intervals $[y_k, y_{k+1}]$ for the function $q(x)p(x)$. If the function $q(x)$ varies slowly enough that it can be assumed to be constant in a narrow interval $[y_k, y_{k+1}]$ (a typical case), then there is exactly one local maximum or minimum in each of the intervals $[y_k, y_{k+1}]$, thus coinciding with the global maximum or minimum in this interval, since the derivative of the polynomial $p(x)$ has degree $n - 1$ and therefore cannot have more than $n - 1$ zeros.

Step 5. Let $a \le z_0 < z_1 < z_2 < ... < z_n \le b$ be a list of alternating positive maxima and negative minima of the function $q(x)p(x)$, which were found in Step 4. Let us consider the values

$$q(z_k)p(z_k) = (-1)^k \varepsilon_k \text{ with } k = 0, 1, 2, ..., n$$

in these points, which, according to Step 4, must be either positive or negative extrema for intervals $[y_k, y_{k+1}]$ with purely positive or purely negative values of the function $q(x)p(x)$. If the condition $\varepsilon_k \approx \text{const}$ is satisfied within the given accuracy, then a polynomial $p(x)$ is obtained, deviating least from zero and satisfying the Chebyshev criterion. At the same time, it follows from the de la Vallée Poussin theorem on the best approximation of a function by polynomials (see the Statement 2) [6, 7] that the exact optimum for problem (2) lies in the range between $\min|\varepsilon_k|$ and $\max|\varepsilon_k|$ (adjusted for the current multiplier before the leading coefficient of the polynomial replacing unity). If the values of ε_k are considerably different from each other, we replace the test points

$$a \le x_0 < x_1 < x_2 < \ldots < x_n \le b$$

with the points

$$a \le z_0 < z_1 < z_2 < \ldots < z_n \le b$$

and return to Step 2.

Step 6. The remaining step is to normalize the polynomial $p(x)$ so that its leading coefficient becomes equal to unity. Since the polynomial $p(x)$ has n roots, and the polynomial itself is not identical to zero, its leading coefficient cannot be zero and, therefore, such normalization can be performed. An additional useful result is applying a list of n real roots of the polynomial $p(x)$ located in the interval $[a, b]$ (found in Step 3) to the polynomial.

The algorithm has a geometrical convergence rate in the following sense: there are numbers $C > 0$ and $0 < \lambda < 1$ for any continuous weight function and for any degree n, for which

$$\max |q(x)p_i(x)| \leq 1 + C\lambda^i,$$

where i is the iteration number.

Convergence of the given algorithm is proved similarly to that of the Remez algorithm in monograph [11]. Numerical experiments established that our algorithm converged very quickly in all practically verified cases.

Fig. 3 illustrates the verification of the algorithm's performance for the case

$$a = 0, b = 1, q(x) = x^3, p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + x^5.$$

Evidently, it is already the third iteration that provides a completely acceptable deviation from zero. The algorithm finally converges in the fifth iteration.

The solution is a polynomial with the coefficients

$$a_0 = -0.018464, a_1 = 0.712942, a_2 = -2.851981,$$

$$a_3 = 4.650795, a_4 = -3.493237$$

The deviation from zero is $5.5 \cdot 10^{-5}$. Fig. 4 shows a comparison between the deviations from zero of the polynomial $q(x)p(x)$ and the first-kind Chebyshev polynomial of the same degree.

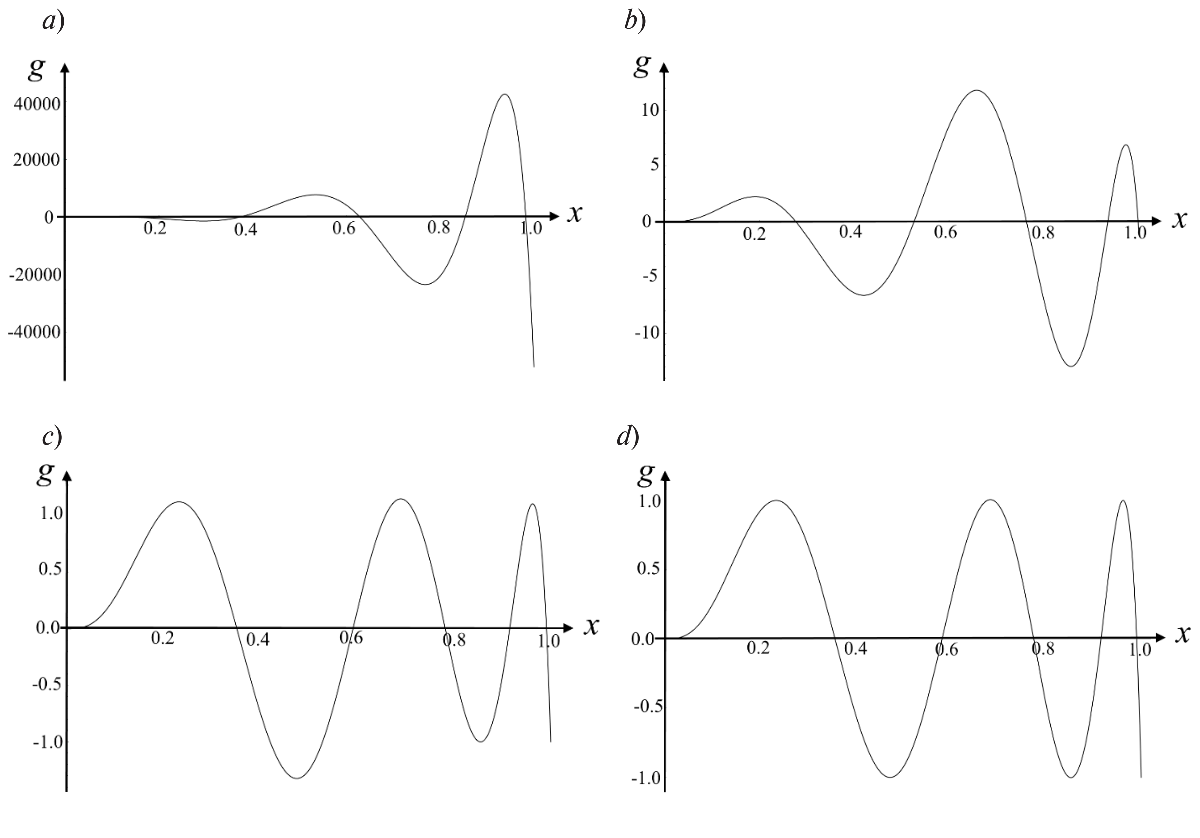

Fig. 3. Iterations $g(x) = q(x)p(x)$ for calculated fifth-degree polynomial $p(x)$ of least deviation from zero in the interval [0, 1] with weight $q(x) = x^3$: initial state after Step 1 of algorithm (*a*), as well as the first (*b*), second (*c*) and third (*d*) iterations

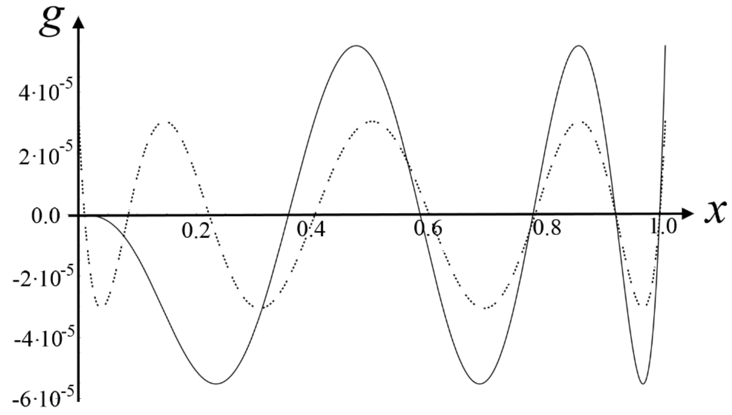

Fig. 4. Oscillations of function $g(x) = q(x)p(x)$ of least deviation from zero in the interval [0, 1] at $n = 5$ and $q(x) = x^3$ (solid line)
Scaled Chebyshev polynomial of degree 5 + 3 = 8, shifted from the interval [-1, +1] to the interval [0, 1], which has 1 as the leading coefficient, is shown by a dashed line

Since the weight function $q(x)$ can vanish at some of the ends of the interval $[a, b]$, the test points selected at Step 1 must lie strictly within the interval $[a, b]$ to ensure that Step 2 can be taken:

$$a < x_0 < x_1 < x_2 < \ldots < x_n < b. \quad (5)$$

Apparently, if $q(a) = 0$ or $q(b) = 0$ and requirement (5) is preserved, the points z_k satisfy the following condition at step 5:

$$a < z_0 < z_1 < z_2 < \ldots < z_n < b,$$

consequently, condition (5) is preserved both in the next iteration and in all subsequent ones. For this reason, it is sufficient to ensure that condition (5) is satisfied at the first step of the algorithm.

To improve the performance of the algorithm at Step 1, it is recommended to choose either the zeros of the first-kind Chebyshev polynomial of degree $n + 1$, or the minima and maxima of the second-kind Chebyshev polynomial of degree n, shifted and scaled from the interval $[-1, +1]$ to the interval $[a, b]$, as the starting points in (5). It is recommended to use the condition $q(x_k)p(x_k) = (-1)^k \cdot [(b - a)/4]^n$ instead of the condition $q(x_k)p(x_k) = (-1)^k$ at Step 2 to avoid unnecessarily large coefficients for the polynomial $p(x)$. This choice of scale corresponds to the oscillations of the first-kind Chebyshev polynomial of degree n (least deviating from zero in the segment $[-1, +1]$ with the weight $q(x) \equiv 1$), recalculated from the interval $[-1, +1]$ to the interval $[a, b]$ and scaled so that the leading coefficient is equal to unity[2].

It is not recommended to use a direct solution of the system of linear equations $q(x_k)p(x_k) = (-1)^k$ with respect to unknown coefficients a_i to find the polynomial $p(x)$, taking the given values at the given points, at Step 2. The matrix of such linear equations for large n (the Vandermonde matrix) is characterized by a high condition number, which is why the solution of the equations is very sensitive to errors in rounding floating-point numbers [15]. The Lagrange formula [15] for a polynomial of degree n, taking given values at given $n + 1$ points, is free from such instability. Aitken's iterative process [15] can be used in practice instead of the explicit Lagrange formula for large degrees of the polynomial, allowing to sequentially add a new degree and a new interpolation point to a pre-existing interpolation

[2] Strictly speaking, this estimate is not entirely accurate, since it does not take into account the form of the weight function $q(x)$. In particular, $n = 5$ and $q(x) = x^3$ for the example considered, so $q(x)p(x)$ is an eighth degree polynomial, and the scale of oscillations for Chebyshev polynomials of degree $n = 8$ instead of $n = 5$ should be considered. The refined estimate of the oscillation amplitude has the form $(4/(b - a))^{-n-m}$, where m is the effective degree of the weight function $q(x)$ (it can be found by approximating $q(x)$ by a minimax norm polynomial with a unit weight function).

polynomial. However, it seems to be best to use one of Newton's finite difference schemes [15] instead of the explicit Lagrange formula at Step 2. Since the algorithm only requires that the value of the polynomial $p(x)$ be calculated at a given point x, there is no need to recalculate the form of Newton's finite differences into a polynomial represented explicitly (1) before the algorithm stops.

We should also note that all calculations should be performed with high accuracy if possible, since rounding errors introduced at intermediate stages can distort the calculation results to an unacceptable degree, causing the algorithm to become unstable.

The Wolfram Mathematica 11 software system was used to perform the calculations [24].

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THE AUTHORS

BERDNIKOV Alexander S.
Institute for Analytical Instrumentation, RAS
31–33, Ivan Chernykh St., St. Petersburg, 198095, Russia
asberd@yandex.ru
ORCID: 0000-0003-0985-5964

SOLOVYEV Konstantin V.
Peter the Great St. Petersburg Polytechnic University
Institute for Analytical Instrumentation, RAS
29 Politechnicheskaya St., St. Petersburg, 195251, Russia
k-solovyev@mail.ru
ORCID: 0000-0003-3514-8577

СВЕДЕНИЯ ОБ АВТОРАХ

БЕРДНИКОВ Александр Сергеевич – *доктор физико-математических наук, ведущий научный сотрудник Института аналитического приборостроения Российской академии наук.*
198095, Россия, г. Санкт-Петербург, ул. Ивана Черных, 31–33, лит. А.
asberd@yandex.ru
ORCID: 0000-0003-0985-5964

СОЛОВЬЕВ Константин Вячеславович – *кандидат физико-математических наук, доцент Высшей инженерно-физической школы Санкт-Петербургского политехнического университета Петра Великого, младший научный сотрудник Института аналитического приборостроения Российской академии наук.*
195251, Россия, г. Санкт-Петербург, Политехническая ул., 29
k-solovyev@mail.ru
ORCID: 0000-0003-3514-8577

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