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THEORETICAL JUSTIFICATION OF NATURAL FREQUENCY IDENTIFICATION IN THE FDD (FREQUENCY DOMAIN DECOMPOSITION) METHOD

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Abstract. This paper is the first to provide a justification for the main criterion of the Frequency Domain Decomposition (FDD) algorithm based on the singular value decomposition of a spectral measured signals' density. The algorithm is used for monitoring the structures such as buildings, bridges, dams, to determine experimentally their state (under operating conditions) without application of vibroexciters. The criterion is used to search for natural frequencies. The justification included the double-ended estimate for the first singular value of the spectral density matrix, the estimate making it possible to prove strictly the criterion's applicability under some fulfilled conditions.

Keywords: frequency domain decomposition, damping coefficient, spectral density matrix, natural mode shape and frequency

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ТЕОРЕТИЧЕСКОЕ ОБОСНОВАНИЕ ИДЕНТИФИКАЦИИ СОБСТВЕННЫХ ЧАСТОТ В МЕТОДЕ FDD (ДЕКОМПОЗИЦИИ В ЧАСТОТНОЙ ОБЛАСТИ)

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Аннотация. В статье впервые приводится обоснование главного критерия метода FDD (декомпозиции в частотной области), основанного на сингулярном разложении матрицы взаимных спектральных плотностей (МВСП) измеренных сигналов. Метод FDD используется при динамическом тестировании сооружений (здания, мосты, плотины) для экспериментального определения их динамических характеристик в условиях нормальной эксплуатации без применения вибровозбудительного оборудования. Указанный критерий применяется для поиска собственных частот. Обоснование включало двустороннюю оценку первого сингулярного значения МВСП, которая позволила математически строго доказать применимость критерия при выполнении определенных условий.

Ключевые слова: декомпозиция в частотной области, коэффициент демпфирования, спектральная плотность сигнала

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Introduction

Experimental evaluation of the dynamic characteristics of unique structures (natural frequencies, natural modes of vibrations, attenuation decrements) is crucial for the construction industry. This practical procedure is also important for hydraulic structures (HS), due to stringent requirements for the safety of their operation, especially in seismic zones. For example, comparison of experimental and calculated dynamic characteristics allows fine-tuning the parameters of finite element models used for computational justification of operational reliability of HS under different types of dynamic loads.

Since the late 1980s, there has been active research into a class of methods allowing to experimentally determine the dynamic characteristics of structures (buildings, bridges, dams) under normal operating conditions. These methods are often grouped under the umbrella term operational modal analysis (OMA). These methods are widely used around the world due to their relatively low cost and great advances made in measuring and recording equipment. A necessary condition for adopting OMA methods is that the operational dynamic impact should have a random, steady behavior, close to white noise.

A widely acclaimed method from the OMA group is commonly known as Frequency Domain Decomposition (FDD). This method is based on singular decomposition of the cross-spectral density matrix (CSDM) of simultaneously performed measurements. It offers the following benefits:

minimum requirements for the number of simultaneous measurements;

formalized criteria for detecting natural frequencies and eigenmodes;

no restriction on ‘proportional damping for the mathematical model of the structure (formally identified eigemodes are complex).

The first of these benefits means that theoretically, any object, even a very complex one, can be examined using only two accelerometers: one stationary (reference), and the other mobile, sequentially moved around the structure.

The FDD method was first introduced in 2000 [1] and further developed in [2–5]. In 2009, it was theoretically reinterpreted in [6]. Some modifications of the method [3, 5, 11] allow to estimate the modal attenuation coefficients. The theoretical foundations of the FDD method are discussed in more detail in monographs [7, 8]. The method has been further improved; interesting modifications are proposed, for example, in [11–14]. The classic version of FDD and several of its subsequent modifications are implemented in the ARTeMIS Modal software package [9], which allows solving the problem of identifying dynamic characteristics based on the data from vibration tests.

The FDD method and its ARTeMIS Modal software have been adopted since 2019 by scientists of the B.E. Vedeneev All-Russian Research Institute of Hydraulic Engineering (VNIIG) at St. Petersburg, Russia. To date, vibration tests were used to determine the dynamic characteristics of the dam at the Bureyskaya HPP; the dam, the basic structure and floor slabs at the Sayano-Shushenskaya HPP; some hydraulic structures (HS) at the Nizhne-Bureyskaya HPP, the dam at the Zeiskaya HPP.

The FDD method is succesfully applied both in model numerical experiments and in practical problems of different levels of complexity. However, no strict justification of the criterion for identifying natural frequencies has been obtained in the literature; the same is true for theoretical estimates of the potential application scope of the method.

The latter circumstance is especially important for HS, since it is often complicated to use OMA methods (and, in particular, FDD) in these structures, as dynamic loads are induced not by a combination of a large number of random technological factors or microseisms (as in the case with public buildings), but rather by purposeful regulation of the operating modes of the structures, for example, the power capacity of hydraulic units in operation.

The goal of our study is to theoretically validate the main criterion of the FDD method for determining the natural vibration frequencies of an object.

In view of this goal, we constructed a double-ended estimate for the first singular value of the cross-spectral density matrix of vibration signals.

Brief description of the basics of the FDD method

To determine the dynamic characteristics of a structure, let us consider the equation of motion for its point masses:

$$\mathbf{M}\ddot{\mathbf{y}}(t) + \mathbf{C}\dot{\mathbf{y}}(t) + \mathbf{K}\mathbf{y}(t) = \mathbf{x}(t), \tag{1}$$

where $\mathbf{x}(t)$ are the loads, $\mathbf{y}(t)$ is the response (N-dimensional vectors); $\mathbf{M}$, $\mathbf{C}$, $\mathbf{K}$ are the matrices of mass, damping and stiffness, respectively.

It was established in monographs [7, 15] that the matrices $\mathbf{M}$, $\mathbf{C}$, $\mathbf{K}$ are symmetric and real; they are matrix constants, i.e., do not depend on time. Their dimension is determined by the number of degrees of freedom N and is equal to $N \times N$. We should also note that the matrix $\mathbf{M}$ is positive definite, and $\mathbf{C}$ and $\mathbf{K}$ are positive semi-definite [7, 15].

Eq. (1) describes the free vibrations of the system in a homogeneous form. Its nontrivial solution can be used to determine the natural frequencies of the damped system ω_{di}, and, in general, the complex eigenmodes $\boldsymbol{\varphi}_i$ (modal vectors).

Because the eigenmodes are linearly independent, the response $\mathbf{y}(t)$ of the system is uniquely decomposed into their linear combination:

$$\mathbf{y}(t) = \boldsymbol{\varphi}_1 \cdot q_1(t) + \boldsymbol{\varphi}_2 \cdot q_2(t) + \ldots = \boldsymbol{\Phi}\mathbf{q}(t), \tag{2}$$

where $\boldsymbol{\Phi}$ is a matrix whose columns are eigenmodes $\boldsymbol{\varphi}_i$, i.e., $\boldsymbol{\Phi} = [\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2, ...]$; $\mathbf{q}(t)$ is a column vector of modal coordinates; t is time.

An approach called the basic frequency model has been used for many years in engineering practice as an initial approximation for identifying the dynamic characteristics (it is called the Basic Frequency Domain in books [7, 8]). The central idea of this approach is that given a small attenuation in the vicinity of a certain natural frequency with the number s, the response of the system is determined mainly by its eigenmode with the same number.

Then the response $\mathbf{y}(t)$ of the system (see Eq. (2)) can be represented as follows:

$$\mathbf{y}(t) \approx \boldsymbol{\varphi}_s \cdot q_s(t). \tag{3}$$

By definition, the correlation function $\mathbf{R}(\tau)$ (this is a matrix function) takes the following form for a stationary process:

$$\mathbf{R}(\tau) = \mathrm{E}\left[\mathbf{y}(t) \cdot \mathbf{y}^T(t+\tau)\right] = \boldsymbol{\varphi}_s \,\mathrm{E}\left[q_s(t) \cdot q_s(t+\tau)\right] \boldsymbol{\varphi}_s{}^T = \mathrm{R}_q(\tau) \boldsymbol{\varphi}_s \boldsymbol{\varphi}_s{}^T, \tag{4}$$

where $R_q(\tau)$ is the autocorrelation function of the modal coordinate, non-negative by definition.

Performing the Fourier transform of the correlation function $\mathbf{R}(\tau)$, we obtain the expression that is of interest to us for the CSDM of the components of the response vector $\mathbf{G}_y(\omega)$, which depends on eigenmodes:

$$\mathbf{G}_y(\omega) = G_q(\omega) \boldsymbol{\varphi}_s \boldsymbol{\varphi}_s^T. \tag{5}$$

where ω is the circular frequency.

The rank of the matrix $\mathbf{G}_y(\omega)$ is equal to unity (since the rank of the product of the matrices does not exceed the ranks of the multipliers), so the matrix has no more than one eigenvalue different from zero. Furthermore, it is apparent that given expression (5), any row or column of the matrix is $\mathbf{G}_y(\omega)$ proportional to the vector of the eigenmode $\boldsymbol{\varphi}_s$.

Let us find the eigenvalues and vectors based on their definition:

$$\mathbf{G}_y(\omega)\mathbf{u} = \lambda \mathbf{u}, \tag{6}$$

$$G_q(\omega)\boldsymbol{\varphi}_K\boldsymbol{\varphi}_K^T\mathbf{u} = G_q(\omega)\boldsymbol{\varphi}_K(\boldsymbol{\varphi}_K^T\mathbf{u}) = G_q(\omega)(\boldsymbol{\varphi}_K^T\mathbf{u})\boldsymbol{\varphi}_K = \lambda\mathbf{u}. \tag{7}$$

It follows from equalities (7) that the eigenvector is equal to the modal vector $\boldsymbol{\varphi}_K$, and the eigenvalue has the following form:

$$\lambda = G_q(\omega)(\boldsymbol{\varphi}_K^{\mathrm{T}}\boldsymbol{\varphi}_K) = G_q(\omega)\left\|\boldsymbol{\varphi}_K\right\|^2. \tag{8}$$

The matrix $\mathbf{G}_y(\omega)$ given by Eq. (5) is obviously symmetric and, since its only nonzero eigenvalue is positive, it can also be argued that it is positive semi-definite. The singular values in such a matrix coincide with its eigenvalues, and the left and right singular vectors are the same.

The matrix $\mathbf{G}_y\mathbf{G}^T_y$ coincides with $\mathbf{G}_y$ up to the coefficient. Indeed,

$$\mathbf{G}_y\mathbf{G}_y^{\mathrm{T}} = \boldsymbol{\varphi}_s\boldsymbol{\varphi}_s^{\mathrm{T}}\,\boldsymbol{\varphi}_s\boldsymbol{\varphi}_s^{\mathrm{T}} = \left\|\boldsymbol{\varphi}_s\right\|^2\boldsymbol{\varphi}_s\boldsymbol{\varphi}_s^{\mathrm{T}} = \left\|\boldsymbol{\varphi}_s\right\|^2\mathbf{G}_y, \tag{9}$$

It follows from the definition of singular value decomposition that the singular vectors of the matrix $\mathbf{G}_y$ coincide with the eigenvectors of the matrix $\mathbf{G}_y\mathbf{G}^T_y$ Therefore, the first singular vector (it corresponds to the maximum, and in our case, only eigenvalue different from zero) is an estimate of the eigenmode $\boldsymbol{\varphi}_s$.

Unfortunately, it is not always possible to represent the response $\mathbf{y}(t)$ in the form (3), i.e., to neglect the influence of other eigenmodes. It was established in [7] that this method is inapplicable for identifying the natural frequencies close in value and the corresponding eigenmodes even in systems with low damping. More accurate methods, in particular FDD, have been developed for this purpose, also reducing the influence of random noise inevitably generated during measurements.

Let us briefly describe the central idea of the FDD method, following [7].

Let the response $\mathbf{y}(t)$ be a linear composition of all modal vectors according to Eq. (2). Let us calculate the correlation function

$$\mathbf{R}(\tau) = \mathrm{E}\left[\mathbf{y}(t)\cdot\mathbf{y}^{\mathrm{T}}(t+\tau)\right], \tag{10}$$

then

$$\mathbf{R}(\tau) = \boldsymbol{\Phi}\mathrm{E}\left[\mathbf{q}(t)\cdot\mathbf{q}^{\mathrm{T}}(t+\tau)\right]\boldsymbol{\Phi}^{\mathrm{T}} = \boldsymbol{\Phi}\mathbf{R}_q\boldsymbol{\Phi}^{\mathrm{T}}. \tag{11}$$

The Fourier transform of the correlation function $\mathbf{R}(\tau)$ gives an expression for the CSDM $\mathbf{G}_y(\omega)$:

$$\mathbf{G}_y(\omega) = \boldsymbol{\Phi}\mathbf{G}_q(\omega)\boldsymbol{\Phi}^{\mathrm{T}}. \tag{12}$$

It follows from the assumption that there is no correlation between the modal coordinates $\mathbf{q}(t)$ [7] that the matrix $\mathbf{G}_q(\omega)$ is diagonal. Since the matrix $\boldsymbol{\Phi}^T$ contains complex elements, its transposition $\boldsymbol{\Phi}^T$ should be replaced by a Hermitian conjugate $\boldsymbol{\Phi}^H$.

Then expression (12) takes the following form:

$$\mathbf{G}_y(\omega) = \boldsymbol{\Phi}\left[g_n{}^2(\omega)\right]\boldsymbol{\Phi}^{\mathrm{H}}, \tag{13}$$

where the diagonal matrix $[g_n{}^2(\omega)]$ contains the autospectral densities of the matrix $\mathbf{G}_q(\omega)$.

The central idea of the FDD method is based on the application of the following singular value decomposition of the matrix:

$$\mathbf{G}_y(\omega) = \mathbf{USU}^{\mathrm{H}} = \mathbf{U}\left[s_n{}^2(\omega)\right]\mathbf{U}^{\mathrm{H}}, \tag{14}$$

where $\mathbf{s}$ is a diagonal matrix of singular numbers arranged in descending order; $\mathbf{U}$ is a matrix consisting of left (right) singular vectors.

The left and right singular vectors of the matrix $\mathbf{G}_y(\omega)$ are the same because this matrix is self-adjoint and positive definite [18].

Comparing expressions (13) and (14), we see that if the eigenvectors composing the matrix $\mathbf{\Phi}$ were mutually orthogonal, then the required modal forms up to a coefficient would be singular vectors of the CSDM at an arbitrary frequency. Since this condition is not fulfilled, we can only expect for an approximate solution to the problem of finding modal vectors and frequencies.

As shown in [7], if the external effect is assumed to be white noise, and the dissipation is small, then the following expression is valid for the matrix $\mathbf{G}_y(\omega)$:

$$\mathbf{G}_y(\omega) = \sum_{m=1}^{M} \frac{c_m \boldsymbol{\varphi}_m \boldsymbol{\varphi}_m^{\mathrm{H}}}{i\omega - \lambda_m} + \frac{c_m \boldsymbol{\varphi}_m \boldsymbol{\varphi}_m^{\mathrm{H}}}{-i\omega - \lambda_m^*} = \mathbf{\Phi} \cdot \mathrm{diag}\left(2\,\mathrm{Re}\left(\frac{c_m}{i\omega - \lambda_m}\right)\right) \cdot \mathbf{\Phi}^{\mathrm{H}}, \tag{15}$$

where $\lambda_m = -\gamma_m + i\omega_{dm}$ (γ_m is the dissipation coefficient, ω_{dm} is the natural frequency taking into account damping); $\boldsymbol{\varphi}_m$ is the eigenmode; $\mathbf{\Phi}$ is the matrix whose columns are the vectors of eigenmodes $\mathbf{\Phi} = [\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2, \ldots, \boldsymbol{\varphi}_M]$; c_m is the positive coefficient; M is the number of eigenmodes taken into account in decomposition (2).

Let us introduce the notations

$$\alpha_m(\omega) = 2\,\mathrm{Re}\left(\frac{c_m}{i\omega - \lambda_m}\right) = \frac{c_m \gamma_m}{(\omega - \omega_{md})^2 + \gamma_m^2}. \tag{16}$$

Then expression (15) can be written as follows:

$$\mathbf{G}_y(\omega) = \mathbf{\Phi} \cdot \mathrm{diag}\left(\alpha_m(\omega)\right) \cdot \mathbf{\Phi}^{\mathrm{H}}, \tag{17}$$

or

$$\mathbf{G}_y(\omega) = \sum_{m=1}^{M} \alpha_m \boldsymbol{\varphi}_m \boldsymbol{\varphi}_m^{\mathrm{H}}. \tag{18}$$

The authors of the FDD method proposed an algorithm for the case when the values of the natural frequencies are not close to each other, based on representation of the matrix $\mathbf{G}_y(\omega)$ in the form (17), which can be summarized as follows.

Step 1. A CSDM $\mathbf{G}_y(\omega)$[1] is calculated for each frequency ω of a given range.

Step 2. A singular value decomposition (SVD) of the matrix $\mathbf{G}_y(\omega)$ is performed at each frequency ω, its first singular value $\sigma_1(\omega)$ is determined and a function for the first singular value of $\sigma_1(\omega)$ depending on the frequency ω is constructed.

Step 3. The values of frequencies $\tilde{\omega}_m$ that correspond to the local maxima of the function $\sigma_1(\omega)$ are found.

Step 4. If singular expansions generate the first singular vectors close to collinear (which is verified using a MAC estimate[2]) in the vicinity of the frequency $\tilde{\omega}_m$, then the frequency $\tilde{\omega}_m$ can be assumed to be a natural frequency, and the first singular vector $\mathbf{u}_1(\omega_m)$ is an estimate of the eigenmode.

Thus, the central idea (referred to as the criterion from now on) of the FDD algorithm is that the first singular number of the matrix $\mathbf{G}_y(\omega_m)$, being a function of frequency, has local maxima near modal frequencies.

This is confirmed by the solutions of model problems and numerous calculations performed on vibration measurements of real objects.

[1] Generally speaking, only those elements of the matrix $\mathbf{G}_y(\omega)$ that can be obtained from signals measured simultaneously are calculated. The algorithm described below can be applied to a complete matrix $\mathbf{G}_y(\omega)$, but the matrices used in practice are selected from the matrix $\mathbf{G}_y(\omega)$ in a specific manner.

[2] MAC is the Modal Assurance Criterion. It is introduced to compare two eigenmodes **a** and **b** by the formula $\mathrm{MAC}(\mathbf{a},\mathbf{b}) = \dfrac{\left|\mathbf{a}^{\mathrm{H}}\mathbf{b}\right|^2}{(\mathbf{a}^{\mathrm{H}}\mathbf{a})(\mathbf{b}^{\mathrm{H}}\mathbf{b})}$.

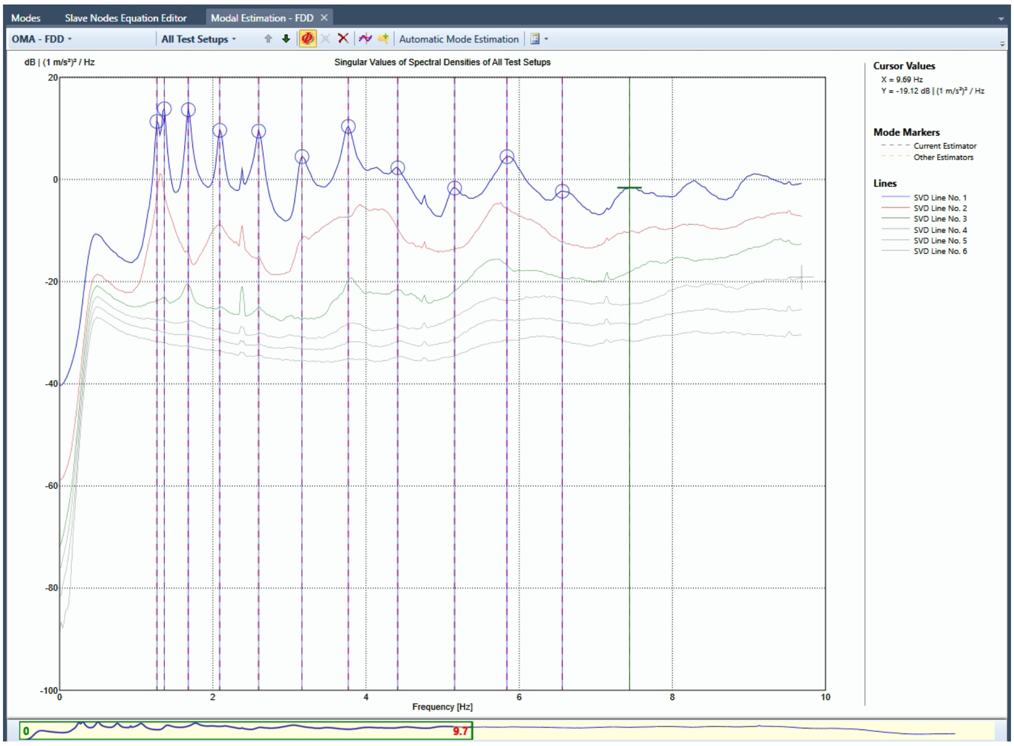

Fig. 1. Averaged frequency dependences of six singular values (6 curves) of the spectral density matrix: the upper curve (blue) corresponds to the first singular number $\sigma_1(\omega)$; the arguments of the local maxima $\sigma_1(\omega)$ (marked with circles) correspond to the natural frequencies of the dam. The data were obtained from measurements at the Sayano-Shushenskaya dam in 2022

Fig. 1 shows a screenshot from the ARTeMIS Modal program with the graphs of the dependences of the first six singular numbers on frequency (in logarithmic units for greater clarity) based on the results of dynamic testing of the Sayano-Shushenskaya HPP dam that we performed in 2022.

We should again emphasize that one of the main goals of dynamic testing is to determine (as accurately as possible) the values of the natural frequencies of structural vibrations. The graphs in Fig. 1 confirm that the method allows to identify the 11 lowest natural frequencies of the Sayano-Shushenskaya HPP dam. The ARTeMIS Modal program provides a special procedure for excluding harmonic components from the procedure for identifying the natural frequencies.

Lack of justification for the criterion of the FDD method

Even though the FDD method is widely used in engineering practice, the mathematical justification of the criterion has not yet been carried out. In other words, the studies on the FDD method do not provide evidence that the function $\sigma_1(\omega)$ has local maxima in the vicinity of natural vibration frequencies.

There is no analytical expression of the first singular number for square matrices of arbitrary dimension. However, as already noted above, the CSDM can be represented as (17) and its structure allows to obtain the necessary estimates.

The coefficients $\alpha_m(\omega)$ are of particular interest to us, since it follows from Eq. (16) that they not only depend on the corresponding natural frequencies and damping coefficients, but also reach their maximum values at natural frequencies.

Indeed, determining the extreme values of the function $\alpha_m(\omega)$, we obtain for $\omega = \omega_{di}$

$$\alpha_i(\omega_{di}) = \frac{c_i}{\gamma_i}. \tag{19}$$

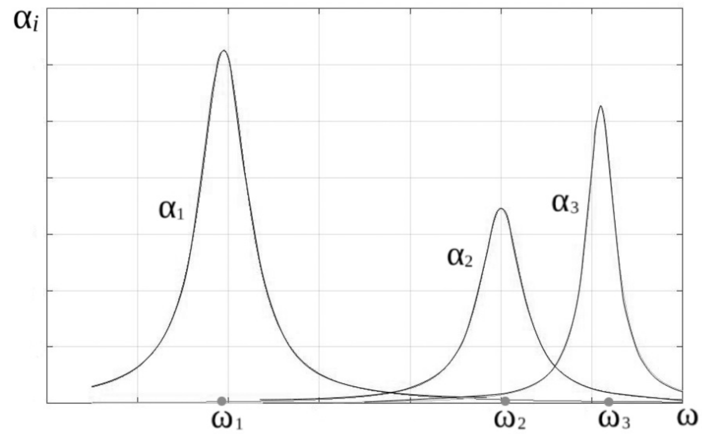

Fig. 2. Functions $\alpha_j(\omega)$ for a system with three degrees of freedom

Fig. 2 shows an example of graphs for the functions $\alpha_j(\omega)$ for a system with three degrees of freedom with small damping coefficients.

We can demonstrate how the values of the damping coefficients and the distance between the natural frequencies affect the result considering the example of the simplest system with three degrees of freedom.

Example of the simplest system with three degrees of freedom. Consider three cases. We define the eigenmodes, dissipation coefficients and natural frequencies as follows.

Case 1. Matrix of modal vectors (eigenvectors)

$$\mathbf{\Phi} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & -1 \end{bmatrix};$$

the values of the damping coefficients are as follows:

$$\gamma_1 = 16,\ \gamma_2 = \gamma_3 = 18,$$

and the values of the natural frequencies are

$$\omega_{d1} = 29.5,\ \omega_{d2} = 52.0,\ \omega_{d3} = 71.0.$$

Case 2. This differs from case 1 only by the value of the second natural frequency, which is $\omega_{d2} = 63.0$.

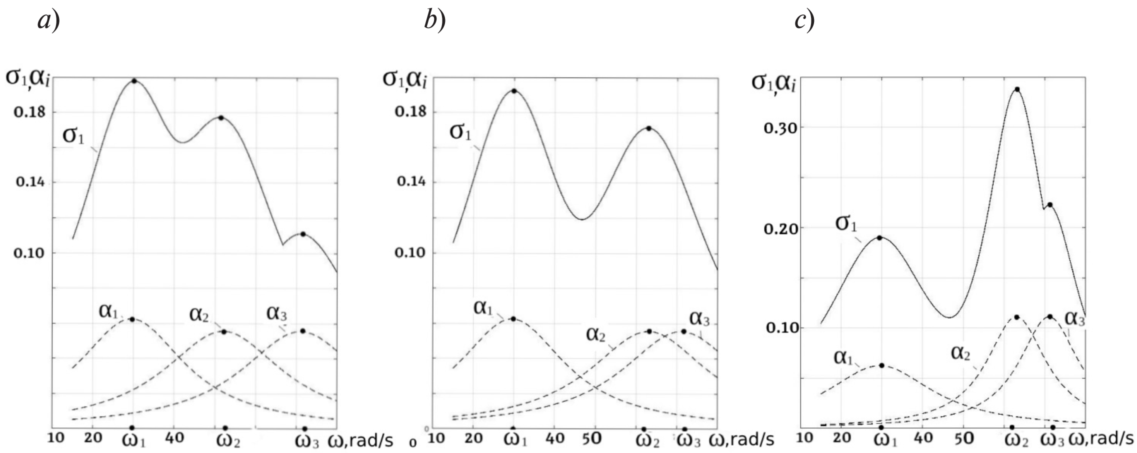

Fig. 3. Functions of first singular value $\sigma_1(\omega)$ and $\alpha_i(\omega)$ (solid and dashed lines, respectively) for cases 1 (*a*), 2 (*b*) and 3 (*c*)

Case 3. This differs from case 2 by the values of the dissipation coefficients: $\gamma_2 = \gamma_3 = 9$.

We construct (with some step) the CSDMs $\mathbf{G}_y(\omega)$ by Eq. (17) and, performing a singular decomposition of these matrices, we construct the curves of the first singular number as function of the circular frequency for each of the cases. We also give graphs of functions $\alpha_m(\omega)$ for all three cases (Fig. 3). Evidently, the maxima of the function of the first singular number in case 1 correspond to the natural frequencies; in case 2, the function $\sigma_1(\omega)$ has only two extrema, and in case 3, where the damping coefficients decrease compared to the previous case, all natural frequencies are determined again.

Thus, some variation of parameter values can produce a qualitatively different result. The behavior of the curves corresponding to the coefficients $\alpha_i(\omega)$ can be clearly seen from the graphs. These functions indicate that it is not only the distance between the eigenmodes that matter, but also the damping coefficients determining the sharpness of the peaks of the functions $\alpha_i(\omega)$.

Construction of double-ended estimate for $\sigma_1(\omega)$

Let us introduce some additional notation:

$$\mathbf{A}^2 = \mathrm{diag}\left(\alpha_m\right). \tag{20}$$

Since the coefficients $c_m > 0$ and $\gamma_m > 0$ [7] in expression (16), the diagonal matrix A^2 consists of real positive elements.

The matrix $\mathbf{A}$ is defined as follows:

$$\mathbf{A} = \mathrm{diag}\left(\sqrt{\alpha_m(\omega)}\right). \tag{21}$$

Eq. (17) can be written in the following form:

$$\mathbf{G}_y(\omega) = \mathbf{\Phi}\mathbf{A}^2\mathbf{\Phi}^{\mathrm{H}}. \tag{22}$$

The matrix $\mathbf{G}_y(\omega)$ is Hermitian (self-adjoint) because

$$\mathbf{G}_y(\omega) = \mathbf{G}_y^{\mathrm{H}}(\omega).$$

Modal vectors $\boldsymbol{\varphi}_m$ can be assumed to be normalized, since the coefficient α_m, according to expression (16), contains a constant c_m that can be complemented with a normalization coefficient.

Below we omit the formulation of the argument ω for functions that depend on it for simplicity's sake, i.e., we write $\mathbf{G}_y$ instead of $\mathbf{G}_y(\omega)$, α_m instead of $\alpha_m(\omega)$, etc.

Let us write the matrix $\mathbf{G}_y$ in coordinate form:

$$\mathbf{G}_y = \begin{bmatrix} \sum_{m=1}^{M}\alpha_m(\varphi_m^{(1)})^2 & \sum_{m=1}^{M}\alpha_m\varphi_m^{(1)}\varphi_m^{(2)} & & \sum_{m=1}^{M}\alpha_m\varphi_m^{(1)}\varphi_m^{(N)} \\ \sum_{m=1}^{M}\alpha_m\varphi_m^{(2)}\varphi_m^{(1)} & \sum_{m=1}^{M}\alpha_m(\varphi_m^{(2)})^2 & \cdots & \sum_{m=1}^{M}\alpha_m\varphi_m^{(2)}\varphi_m^{(N)} \\ \cdots & \cdots & & \cdots \\ \sum_{m=1}^{M}\alpha_m\varphi_m^{(N)}\varphi_m^{(1)} & \sum_{m=1}^{M}\alpha_m\varphi_m^{(N)}\varphi_m^{(2)} & & \sum_{m=1}^{M}\alpha_m(\varphi_m^{(N)})^2 \end{bmatrix}. \tag{23}$$

The first singular value of this matrix coincides with its spectral radius (this statement will be proved below). However, the structure of this matrix is rather complex to obtain estimates of the spectral radius, since its elements contain separate components of modal vectors.

Let us confirm that this matrix is reduced to a simpler form preserving the spectrum by means of some operation.

Let us consider the following matrix:

$$\mathbf{K} = (\mathbf{A}\mathbf{\Phi}^{\mathrm{H}})(\mathbf{\Phi}\mathbf{A}) \tag{24}$$

and confirm that the matrices $\mathbf{G}_y$ and $\mathbf{K}$ have the same nonzero eigenvalues.

For this purpose, we prove the following auxiliary statement.

Statement. *Let* **U** *and* **V** be *some rectangular matrices of dimension n × m. Then the nonzero eigenvalues of the matrices* $\mathbf{UV}^{\mathrm{H}}$ *and* $\mathbf{V}^{\mathrm{H}}\mathbf{U}$ *coincide.*

Proof. Let some nonzero number λ be the eigenvalue of the matrix $\mathbf{UV}^{\mathrm{H}}$, i.e., there exists a nonzero vector $\mathbf{u}$ such that

$$(\mathbf{UV}^{\mathrm{H}})\mathbf{u} = \lambda\mathbf{u}. \tag{25}$$

We multiply both parts (25) by $\mathbf{V}^{\mathrm{H}}$ on the left and, using the associativity property of matrix multiplication, we obtain the equality:

$$\mathbf{V}^{\mathrm{H}}\mathbf{U}(\mathbf{V}^{\mathrm{H}}\mathbf{u}) = \lambda(\mathbf{V}^{\mathrm{H}}\mathbf{u}). \tag{26}$$

Since the number λ is different from zero and the vector $\mathbf{u}$ is nonzero, then the vector $\mathbf{V}^{\mathrm{H}}\mathbf{u}$ is nonzero as well (this is evident if we scalar multiply equality (26) by itself), which means that the number λ also turns out to be the eigenvalue of the matrix $\mathbf{V}^{\mathrm{H}}\mathbf{u}$ (by definition of the eigenvalue and vector).

Statement is proven.

If we assume that

$$\mathbf{U} = \mathbf{V} = \mathbf{\Phi A}, \tag{27}$$

that is, because the following equality holds true:

$$\mathbf{G}_y(\omega) = \mathbf{\Phi A}^2\mathbf{\Phi}^{\mathrm{H}} = (\mathbf{\Phi A})(\mathbf{A\Phi}^{\mathrm{H}}), \tag{28}$$

it follows from the statement proved above that the matrices

$$\mathbf{G}_y(\omega) = (\mathbf{\Phi A})(\mathbf{A\Phi}^{\mathrm{H}}) \text{ and } \mathbf{K} = (\mathbf{A\Phi}^{\mathrm{H}})(\mathbf{\Phi A})$$

have the same eigenvalues, different from zero.

By virtue of the combinational law, the matrix $\mathbf{K}$ can be represented as

$$\mathbf{K} = \mathbf{A}(\mathbf{\Phi}^{\mathrm{H}}\mathbf{\Phi})\mathbf{A}. \tag{29}$$

It takes the following coordinate form:

$$\mathbf{K} = \begin{bmatrix} \alpha_1 & \sqrt{\alpha_1\alpha_2}(\varphi_1,\varphi_2) & & \sqrt{\alpha_1\alpha_M}(\varphi_1,\varphi_M) \\ \sqrt{\alpha_1\alpha_2}(\varphi_2,\varphi_1) & \alpha_2 & \dots & \sqrt{\alpha_2\alpha_M}(\varphi_2,\varphi_M) \\ \dots & \dots & & \dots \\ \sqrt{\alpha_1\alpha_M}(\varphi_M,\varphi_1) & \sqrt{\alpha_2\alpha_M}(\varphi_M,\varphi_2) & & \alpha_M \end{bmatrix}. \tag{30}$$

The matrix $\mathbf{K}$ composed of scalar products of a vector system $\sqrt{\alpha_i}\boldsymbol{\varphi}_i$, taking into account that $\|\varphi_i\| = 1$, is a Gram matrix [17], which is known to be Hermitian. Since the matrix $\mathbf{K}$ is constructed relative to modal vectors, and modal vectors are linearly independent, it turns out to be strictly positive definite.

From this we can conclude that the singular numbers are identical to the eigenvalues and, consequently, the first singular number is equal to the spectral radius of the matrix $\mathbf{K}$.

According to the above, the matrices $\mathbf{G}_y$ (dimension $N \times N$) and $\mathbf{K}$ (dimension $M \times M$) have the same eigenvalues different from zero; however, since the dimensions of the matrices differ, the matrix $\mathbf{G}_y$ can also have zero eigenvalues in the case when $N > M$. Therefore, it is positive semi-definite. Because the matrix $\mathbf{G}_y$ is positive semi-definite and self-conjugate, it follows that the singular numbers are identical to the eigenvalues and, consequently, as in the case of the matrix $\mathbf{K}$, the first singular number is equal to the spectral radius.

According to the theorem on singular number estimates [17], the first singular value of the Hermitian matrix **P** cannot be less than the modulus of its maximal diagonal element:

$$\sigma_1 \geq \max_{1\leq i\leq M} \left|\mathbf{P}_{ii}\right|, \tag{31}$$

where σ_1 is the first singular number.

We thus obtain the lower-bound estimate of the first singular value $\sigma_1(\omega)$ of the matrix **K**, and, consequently, the matrix $\mathbf{G}_y$:

$$\sigma_1 \geq \max_{1\leq i\leq M} \alpha_i. \tag{32}$$

To obtain an upper-bound estimate of the first singular value $\sigma_1(\omega)$ of the matrix $\mathbf{G}_y$, we consider the matrix $\mathbf{T} = (\mathbf{\Phi}^{\mathrm{H}}\mathbf{\Phi})\mathbf{A}^2$ and confirm that the spectrum of this matrix coincides with the spectrum of the matrix **K,** and, therefore, the nonzero eigenvalues of the matrices **T** and $\mathbf{G}_y(\omega)$ also coincide.

Let us introduce the following notations:

$$\mathbf{C} = \mathbf{\Phi}^H\mathbf{\Phi}. \tag{33}$$

Then the matrix **K** can be written as

$$\mathbf{K} = \mathbf{ACA}, \tag{34}$$

and the matrix **T** as

$$\mathbf{T} = \mathbf{CA}^2. \tag{35}$$

Now let λ be the eigenvalue, and let **u** be the corresponding eigenvector of the matrix **K**, i.e., $\mathbf{Ku} = \lambda\mathbf{u}$, and then

$$(\mathbf{ACA})\mathbf{u} = \lambda\mathbf{u}. \tag{36}$$

Multiplying equality (32) in the left-hand side by $\mathbf{A}^{-1}$, we obtain:

$$\mathbf{CAu} = \lambda\mathbf{A}^{-1}\mathbf{u}. \tag{37}$$

This implies equalities of the form

$$\mathbf{CA}(\mathbf{AA}^{-1})\mathbf{u} = \mathbf{CA}^2(\mathbf{A}^{-1}\mathbf{u}) = \mathbf{T}(\mathbf{A}^{-1}\mathbf{u}) = \lambda\mathbf{A}^{-1}\mathbf{u}. \tag{38}$$

Thus, the eigenvalues of the matrix $\mathbf{T} = \mathbf{CA}^2$ coincide with the eigenvalues of the matrix $\mathbf{K} = \mathbf{ACA}$.

Let us write the matrix **T** in coordinate form:

$$\mathbf{T} = \begin{bmatrix} \alpha_1 & \alpha_2(\varphi_1,\varphi_2) & & \alpha_M(\varphi_1,\varphi_M) \\ \alpha_1(\varphi_2,\varphi_1) & \alpha_2 & \cdots & \alpha_M(\varphi_2,\varphi_M) \\ \ldots & \ldots & & \ldots \\ \alpha_1(\varphi_M,\varphi_1) & \alpha_2(\varphi_M,\varphi_2) & & \alpha_M \end{bmatrix}. \tag{39}$$

Since the spectrum of the matrices **T** coincides with the spectrum **K**, all eigenvalues are positive, so the matrix is positive definite.

The following relation between norms holds true in finite-dimensional spaces [18]:

$$\rho(\mathbf{T}) \leq \left\|\mathbf{T}\right\|_\infty, \tag{40}$$

where $\rho(\mathbf{T})$ is the spectral radius (maximum eigenvalue) of the matrix **T**; $\|\mathbf{T}\|_\infty$ is the matrix norm taking the form

$$\|\mathbf{T}\|_{\infty} = \max_{1\le i\le M} \sum_{j=1}^{M} \left|\mathbf{T}_{ij}\right|. \tag{41}$$

Expressions (39)–(41) imply the following estimate of the spectral radius of the matrix **T**:

$$\rho \le \max_{1\le i\le M} \sum_{j=1}^{M} \alpha_j \left|(\varphi_i \varphi_j)\right|. \tag{42}$$

As established above, the nonzero eigenvalues of the matrices $\mathbf{G}_y$ and **T** coincide, and the first singular number of the matrix $\mathbf{G}_y$ coincides with the spectral radius. Therefore, the upper-bound estimate of the first singular value $\sigma_1(\omega)$ of the matrix $\mathbf{G}_y$ is also determined by the expression (38):

$$\sigma_1 \le \max_{1\le i\le M} \sum_{j=1}^{M} \alpha_j \left|(\varphi_i \varphi_j)\right|. \tag{43}$$

Since the vectors φ_i are normalized, the scalar product $(\varphi_i \varphi_j)$ is the cosine of the angle between the vectors φ_i and φ_j, i.e., $|(\varphi_i \varphi_j)| \le 1$, and the following notation is valid:

$$\sigma_1 \le \max_{1\le i\le M} \sum_{j=1}^{M} \alpha_j \left|(\varphi_i \varphi_j)\right| \le \sum_{k=1}^{M} \alpha_k = \mathrm{Tr}(\mathbf{G}_y). \tag{44}$$

Combining the lower-bound (32) and upper-bound (44) estimates, we write a double-ended estimate for the first singular value $\sigma_1(\omega)$ of the matrix $\mathbf{G}_y$:

$$\max_{1\le i\le M} \alpha_i \le \sigma_1 \le \max_{1\le i\le M} \sum_{j=1}^{M} \alpha_j \left|(\varphi_i \varphi_j)\right| \le \sum_{k=1}^{M} \alpha_k = \mathrm{Tr}(\mathbf{G}_y). \tag{45}$$

If the modal vectors are mutually orthogonal, then the following estimate $\sigma_1(\omega)$ follows from (45):

$$\max_{1\le i\le M} \alpha_i \le \sigma_1(\omega) \le \max_{1\le i\le M} \alpha_i. \tag{46}$$

Evidently, this means that

$$\sigma_1(\omega) = \max_{1\le i\le M} \alpha_i. \tag{47}$$

Importantly, these estimates are valid for the considered functions $\sigma_1(\omega)$, $\alpha_m(\omega)$ and $\mathrm{Tr}(\mathbf{G}_y(\omega))$ at all frequencies, and not only in the vicinity of natural frequencies.

Let us introduce the notations

$$d_i = \min_{j\ne i} \left|\omega_{di} - \omega_{dj}\right|. \tag{48}$$

Then the following relations are fulfilled for the frequency ω_{di}, for all $j \ne 1$:

$$\alpha_j(\omega_{di}) \le \frac{c_j \gamma_j}{d_i^2 + \gamma_j^2} = \frac{c_j / \gamma_j}{(d_i / \gamma_j)^2 + 1}. \tag{49}$$

Let us compare relations (49) with the formula $\alpha_i(\omega_{di}) = c_i/\gamma_i$. We can see from this comparison that if $d_i / \gamma_j \gg 1$ for all $j = 1, 2, \ldots, M$, then

$$\alpha_i(\omega_{di}) \gg \alpha_j(\omega_{di}), \tag{50}$$

and therefore, if condition (50) is satisfied, then both functions $\max\limits_{1\le i\le M}\sum\limits_{j=1}^{M}\alpha_j\left|(\boldsymbol{\varphi}_i\boldsymbol{\varphi}_j)\right|$ and $\mathrm{Tr}(\mathbf{G}_y)$ approach the function $\max\limits_{1\le i\le M}\alpha_j$, and, consequently, we can claim that $\sigma_1(\omega)$, which is the function of the first singular number in the SVD decomposition of the matrix $\mathbf{G}_y(\omega)$, also approaches this function if condition (50) is satisfied, with its maxima then located near the modal frequencies.

However, the values of the function $\max\limits_{1\le i\le M}\sum\limits_{j=1}^{M}\alpha_j\left|(\boldsymbol{\varphi}_i\boldsymbol{\varphi}_j)\right|$ at any of the frequencies are obviously much closer to the function $\max\limits_{1\le i\le M}\alpha_j$ than to the function $\sum\limits_{k=1}^{M}\alpha_k = \mathrm{Tr}(\mathbf{G}_y)$, due to the fact that $|(\varphi_i\varphi_j) < 1$.

The graphs of the function $\sigma_1(\omega)$ and the functions of the lower and upper-bound estimates (45) are shown in Fig. 4 for a system with three degrees of freedom to illustrate this statement.

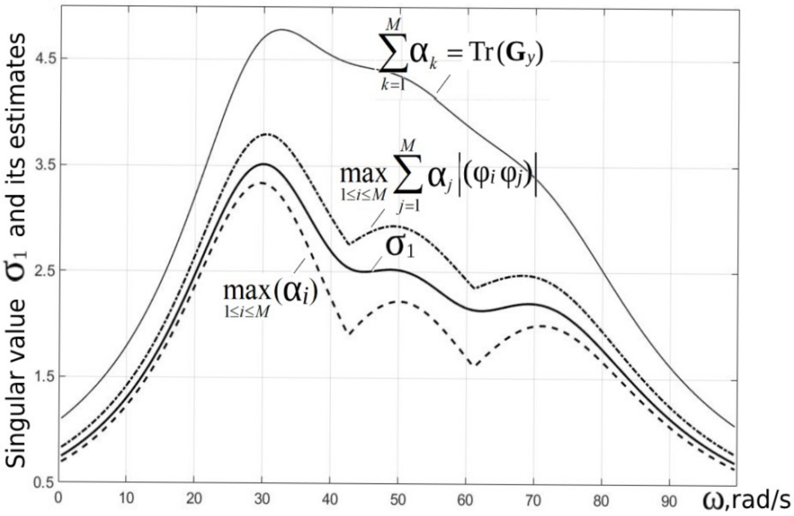

Fig. 4. Comparison of first singular value of spectral density matrix $\sigma_1(\omega)$ with its lower and upper-bound estimates by Eq. (45) as functions of the frequency ω

Conclusion

We constructed for the first time a double-ended estimate for the first singular value of the cross-spectral density matrix of vibration responses in a linear mechanical system with many degrees of freedom. This estimate serves as a justification for the main criterion of the frequency domain decomposition (FDD) method aimed at searching for natural vibration frequencies based on the results of vibration measurements.

The study carried out can be used to further improve the FDD method, analyze the scope of its applicability for mechanical systems with significant damping, and compare the FDD method with other methods of operational modal analysis (OMA) in identifying the dynamic characteristics of structures.

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