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FREE VIBRATIONS OF A RECTANGULAR PLATE WITH CLAMPED OPPOSITE EDGES (A CFCF-PLATE)

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Abstract. The paper proposes an iterative method for finding the natural frequencies of symmetric oscillations for a rectangular CFCF plate using two hyperbolic-trigonometric series in two coordinates. In this case, a resolving homogeneous infinite system of linear equations which contains the desired natural frequency as a parameter has been obtained. Instead of deriving and solving the frequency equation, it was proposed to use the sequential search for the desired frequency values (the "shooting method") in combination with the method of successive approximations. Numerical results were found for the first three symmetric eigenforms. The influence of the number of series terms and the number of iterations on the accuracy of the results was investigated. The obtained results were compared with those of other authors and the published experimental data.

Keywords: rectangular CFCF-plate, oscillations, natural frequency, hyperbolic-trigonometric series, iterative process

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СВОБОДНЫЕ КОЛЕБАНИЯ ПРЯМОУГОЛЬНОЙ ПЛАСТИНЫ С ЗАЩЕМЛЕННЫМИ ПРОТИВОПОЛОЖНЫМИ КРАЯМИ (CFCF-ПЛАСТИНА)

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Аннотация. В работе предложен итерационный метод отыскания собственных частот симметричных колебаний для прямоугольной CFCF-пластины с помощью двух гиперболо-тригонометрических рядов по двум координатам. При этом получена разрешающая однородная бесконечная система линейных алгебраических уравнений, которая содержит в качестве параметра искомую собственную частоту. Вместо вывода и решения частотного уравнения предложено использовать метод перебора значений частоты («метод стрельбы») в сочетании с методом последовательных приближений. Численные результаты получены для первых трех симметричных собственных форм. Исследовано влияние на точность результатов количества членов рядов и числа итераций. Проведено сравнение с результатами других авторов и опубликованными экспериментальными данными.

Ключевые слова: прямоугольная CFCF-пластина, колебания, собственная частота, гиперболо-тригонометрический ряд, итерационный процесс

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Introduction

Rectangular plates with two clamped opposite edges and two free ones, called CFCF plates, are widely used in diverse fields of industry and in nanotechnology.

An important component of strength calculations in plates is taking into account their potential transverse vibrations during operation, which primarily means that the spectrum of their natural frequencies is to be determined. This information is necessary to find the dynamic stresses induced by variable external loads. External influences (excitations) on the plate can be periodic in nature with different frequencies. If the excitation frequency coincides with the natural frequency of the plate, the resonance phenomenon occurs, which can lead to rapid fracture of the plate. Given a varying excitation frequency, not only the frequency of the first (main) tone of free vibrations, but also the frequencies of several subsequent overtones, for which resonance can also occur.

An exact solution to the problem of free vibrations of a rectangular plate has been obtained only for a simply supported plate (SSSS plate) [1] and a plate whose two parallel faces are supported, and the other two are clamped or free (SCSC and SFSF plates). S here stands for supported, that is, a hinge-supported edge (the other notations are given above). The known approximate solutions [2–12] for the CFCF plate considered here require analysis of the accuracy of the numerical results obtained.

The Ritz (Rayleigh–Ritz) method and its various modifications were used in [2–5, 8, 9], to solve the problem. The American researcher Arthur W. Leissa [2] represented the required deflection function as 36 terms: combinations of trigonometric and hyperbolic functions. Monterubbio and Ilanko [3] used the Rayleigh–Ritz method in combination with the penalty function method. The solution contained up to 40 terms that were combinations of low-order polynomials and trigonometric functions. Narita and Innami [4, 5] used polynomials for different types of boundary conditions in combination with the Pylya theory. The study by Liu and Banerjee [6] applied the analytical method of spectral dynamic stiffness with two types of one-dimensional modified Fourier series for cases of plane stresses and plane deformations. Wang and Werley [7] used the Kantorovich–Krylov variational method. The required deflection function was chosen as a linear combination of trigonometric and hyperbolic functions. Liew et al. [8] developed a computationally efficient method for the orthogonal plate function in the Rayleigh–Ritz procedure. The orthogonal plate function is constructed using the Gram–Schmidt orthogonality relation. Mizusawa [9] constructed the solution using B-splines.

Kshirsagar and Bhaskar [10] introduced a form of superposition method with the solution in trigonometric series. The study by Eisenberger and Deutsch [11] used special combinations of trigonometric and hyperbolic sine and cosine functions. They are intended to produce an exact solution to the problem for many types of boundary conditions. Li et al. [12] proposed a new symplectic superposition method (modification of the Euler method for the Hamilton equations) for solving the problem of vibrations of rectangular plates.

Singal et al. [13] conducted experimental studies to obtain the natural vibration frequencies of rectangular plates with different types of boundary conditions.

Natural vibration frequencies were found in [2–13] for rectangular plates with hinged, clamped and free edges in various combinations. The first two frequencies of the symmetric solution for the CFCF plate from these results are given in Table for comparison with those obtained in our paper.

The sequential search for the frequency parameter applied in this study combined with the iterative process of obtaining nonzero solutions of an infinite homogeneous system of linear algebraic equations with respect to unknown coefficients of hyperbolic trigonometric series has been successfully adopted in our earlier papers to solve a number of problems on oscillations and stability of rectangular plates [14, 15].

Problem statement

Consider a rectangular plate of constant thickness h with plan dimensions $a \times b$. We set the origin of the rectangular coordinate system XOY in the center of the plate. The coordinate axes are drawn parallel to its edges. Let the edges $x = \pm a/2$ be clamped, and the edges $y = \pm b/2$ be free. Adopting dimensionless coordinates $x = X/b$, $y = Y/b$, we obtain the dimensions of the plate $\gamma \times 1$, where $\gamma = a/b$ is the ratio of the sides of the plate (Fig. 1).

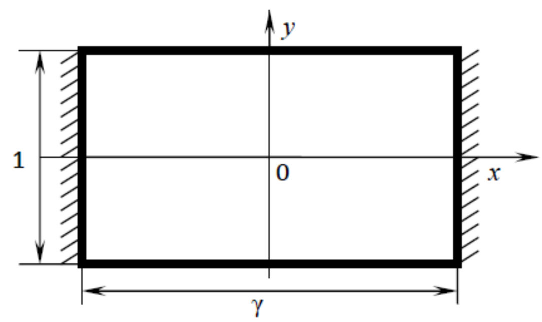

Fig. 1. Schematic for the problem statement: CFCF-plate of size $a \times b \times h$, or $\gamma \times 1 \times h$, where $\gamma = a/b$

The initial conditions of the problem are formulated as follows in the general case (initial field of displacements and velocities):

$$W\big|_{t=0} = W_0(X,Y),\ \frac{\partial W}{\partial t}\big|_{t=0} = V_0(X,Y), \tag{1}$$

where W_0 is the initial deflection of the points in the plate's mid-surface, V_0 are their initial velocities in the direction of the normals, t is time.

A zero-velocity field is often set in the initial conditions (for example, if a stationary uniform transverse load is applied, and then instantly removed). The plate then makes transverse vibrations.

If there are no resistance forces and perturbations, then the vibrations are free undamped (natural vibrations).

The differential equation of natural vibrations of the plate has the form [1]:

$$D\nabla^2\nabla^2 W + \rho h \frac{\partial^2 W}{\partial t^2} = 0, \tag{2}$$

where D is the cylindrical stiffness of the plate, $D = \dfrac{Eh^3}{12(1-\nu^2)}$ (E is Young's modulus, ν is Poisson's ratio); ρ is the density of the plate material; $\nabla^2\nabla^2$ is the biharmonic operator, $\nabla^2\nabla^2 = \dfrac{\partial^4}{\partial X^4} + 2\dfrac{\partial^4}{\partial X^2 \partial Y^2} + \dfrac{\partial^4}{\partial Y^4}$.

In dimensionless coordinates, Eq. (2) are rewritten as follows:

$$\nabla^4 W + \eta^2 \frac{\partial^2 W}{\partial t^2} = 0, \tag{3}$$

where $\eta^2 = \dfrac{\rho h b^4}{D}$, $\nabla^4 = \dfrac{\partial^4}{\partial x^4} + 2\dfrac{\partial^4}{\partial x^2 \partial y^2} + \dfrac{\partial^4}{\partial y^4}$.

The relative dimensions of the plate lie in the intervals

$$-\gamma/2 \le x \le \gamma/2, \ \ -1/2 \le y \le 1/2.$$

The boundary conditions take the following form on the clamped edges $x = \pm\gamma/2$:

$$W = 0, \ \ \frac{\partial W}{\partial x} = 0, \tag{4}$$

and the following form on the free edges $y = \pm 1/2$:

$$\begin{gathered} M_y = \frac{\partial^2 W}{\partial y^2} + \nu \frac{\partial^2 W}{\partial x^2} = 0, \\ V_y = \frac{\partial^3 W}{\partial y^3} + (2-\nu)\frac{\partial^3 W}{\partial x^2 \partial y} = 0. \end{gathered} \tag{5}$$

The problem consists in finding the deflection function $W(x,y,t)$ satisfying differential equation (3), initial conditions (1) and boundary conditions (4), (5).

Construction of the solution

According to the Fourier method, the required deflection function is represented as a product of two functions: of time and of the coordinates of the mid-surface:

$$W(x, y, t) = w^*(t)\, w(x, y), \tag{6}$$

The boundary conditions for the coordinate function are similar to conditions (4), (5), namely,

$$\text{for } x = \pm\gamma/2: w = 0, \ \partial w/\partial x = 0, \tag{7}$$

$$\text{for } y = \pm 1/2 : w = 0, \ \partial w / \partial x = 0,$$

$$M_y = \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} = 0, \ V_y = \frac{\partial^3 w}{\partial y^3} + (2-\nu)\frac{\partial^3 w}{\partial x^2 \partial y} = 0. \tag{8}$$

The time function is represented in the following form [1]:

$$w^*(t) = c_1 \cos pt + c_2 \sin pt, \tag{9}$$

where p is the required vibration frequency of the plate; c_1, c_2 are arbitrary constants, which are determined from initial conditions (1).

Substituting expression (9) into Eq. (6), and then (6) into Eq. (3), we obtain a differential equation that the coordinate function of deflections must satisfy:

$$\nabla^2\nabla^2 w(x,y) - p^2\eta^2 w(x,y) = 0. \tag{10}$$

We denote the relative frequency of natural vibrations by $\Omega = p\eta$. Eq. (10) then takes the form

$$\nabla^2\nabla^2 w(x,y) - \Omega^2 w(x,y) = 0. \tag{11}$$

Symmetric solution (SS-shape). We will search for the function $w(x,y)$ in the form of the sum of two series:

$$w(x,y) = w_1(x,y) + w_2(x,y) =$$

$$= \sum_{k=1}^{\infty} (-1)^k A_k \operatorname{ch}(\beta_k x)\cos(\lambda_k y) + \sum_{s=1,3,\ldots}^{\infty} (-1)^{\tilde{s}} C_s \operatorname{ch}(\xi_s y)\cos(\mu_s x). \tag{12}$$

Here the coefficients A_k, β_k, C_s, ξ_s are to be determined; $\lambda_k = 2\pi k$, $\mu_s = \pi s/\gamma$, $\tilde{s} = (s+1)/2$.

The first eigenmode should be symmetric (more precisely, symmetric-symmetric (SS)), to correspond to the shape of a curved surface under the action of a uniform transverse load (this is why even coordinate functions were chosen).

Other eigenmodes, antisymmetric (AA) and mixed (AS and SA) are obtained with the respective combinations of even and odd terms. This means that the three above cases should be also considered in addition to representation (12).

Thus, function (12) allows finding only symmetric natural frequencies, including the fundamental frequency.

We require that both functions satisfy differential equation (11); then, to determine the coefficients β_k and ξ_s, we obtain the equations

$$\left(\beta_k^2 - \lambda_k^2\right)^2 = \Omega^2, \ \left(\xi_s^2 - \mu_s^2\right)^2 = \Omega^2;$$

we then find pairs of roots depending on an unknown frequency p:

$$\beta_k = \sqrt{\lambda_k^2 + \Omega}, \ \overline{\beta}_k = \sqrt{\lambda_k^2 - \Omega}, \tag{13}$$

$$\xi_s = \sqrt{\mu_s^2 + \Omega}, \ \overline{\xi}_s = \sqrt{\mu_s^2 - \Omega}. \tag{14}$$

In view of the expressions obtained, the functions included in expression (12) are rewritten in the following form:

$$w_1(x,y) = \sum_{k=1}^{\infty} (-1)^k \left(A_k \operatorname{ch}(\beta_k x) + B_k \operatorname{ch}(\overline{\beta}_k x)\right)\cos(\lambda_k y),$$

$$w_2(x,y)=\sum_{s=1,3,...}^{\infty}(-1)^{\tilde{s}}\left(C_s\,\mathrm{ch}(\xi_s y)+D_s\,\mathrm{ch}(\overline{\xi}_s y)\right)\cos(\mu_s x). \quad (15)$$

Notice that the function $w_2(x, y)$ satisfies the condition that there are no deflections on the faces $x = \pm\gamma/2$. We also require for the function $w_1(x, y)$ to have zero deflections on these faces. Then the following equality should hold true:

$$A_k\,\mathrm{ch}\frac{\beta_k\gamma}{2}+B_k\,\mathrm{ch}\frac{\overline{\beta}_k\gamma}{2}=0, \quad (16)$$

and the coefficients B_k can be expressed in terms of the coefficients A_k:

$$B_k=-A_k\frac{\mathrm{ch}\beta_k^*}{\mathrm{ch}\overline{\beta}_k^*}, \quad (17)$$

where $\beta_k^*=\beta_k\gamma/2,\ \overline{\beta}_k^*=\overline{\beta}_k\gamma/2.$

The function $w_1(x, y)$, in turn, satisfies the condition for the absence of shear forces on the faces $y = \pm1/2$. We require for the function $w_2(x, y)$ to also satisfy this condition. Then, omitting the summation sign, we obtain:

$$C_s\xi_s\left[\xi_s^2-(2-\nu)\mu_s^2\right]\mathrm{sh}\xi_s^*+D_s\overline{\xi}_s\left[\overline{\xi}_s^2-(2-\nu)\mu_s^2\right]\mathrm{sh}\overline{\xi}_s^*=0, \quad (18)$$

where $\xi_s^*=\xi_s/2,\ \overline{\xi}_s^*=\overline{\xi}_s/2$.

Then,

$$D_s=-\frac{\xi_s\left[\xi_s^2-(2-\nu)\mu_s^2\right]\mathrm{sh}\xi_s^*}{\overline{\xi}_s\left[\overline{\xi}_s^2-(2-\nu)\mu_s^2\right]\mathrm{sh}\overline{\xi}_s^*}C_s. \quad (19)$$

So, the required deflection function (12) takes the form:

$$w(x,y)=\sum_{k=1}^{\infty}(-1)^k A_k\left(\mathrm{ch}(\beta_k x)-\frac{\mathrm{ch}\beta_k^*}{\mathrm{ch}\overline{\beta}_k^*}\mathrm{ch}(\overline{\beta}_k x)\right)\cos(\lambda_k y)+$$
$$+\sum_{s=1,3,...}^{\infty}(-1)^{\tilde{s}}C_s\left(\mathrm{ch}(\xi_s y)-\frac{\xi_s\left[\xi_s^2-(2-\nu)\mu_s^2\right]\mathrm{sh}\xi_s^*}{\overline{\xi}_s\left[\overline{\xi}_s^2-(2-\nu)\mu_s^2\right]\mathrm{sh}\overline{\xi}_s^*}\mathrm{ch}(\overline{\xi}_s y)\right)\cos(\mu_s x). \quad (20)$$

This function now contains only two sequences of unknown coefficients: A_k and C_s.

We require for deflection function (20) to satisfy the two remaining boundary conditions (the second condition (7) and the first condition (8)). Then, we obtain two equations for the angles of rotation of the clamped edges and bending moments on the free edges:

$$\sum_{k=1}^{\infty}(-1)^k A_k\,\mathrm{ch}\beta_k^*\left(\beta_k\,\mathrm{th}\beta_k^*-\overline{\beta}_k\,\mathrm{th}\,\overline{\beta}_k^*\right)\cos(\lambda_k y)+$$
$$+\sum_{s=1,3,...}^{\infty}\mu_s C_s\left(\mathrm{ch}(\xi_s y)-\frac{\xi_s\left[\xi_s^2-(2-\nu)\mu_s^2\right]\mathrm{sh}\xi_s^*}{\overline{\xi}_s\left[\overline{\xi}_s^2-(2-\nu)\mu_s^2\right]\mathrm{sh}\overline{\xi}_s^*}\mathrm{ch}(\overline{\xi}_s y)\right)=0,$$
$$\sum_{k=1}^{\infty}A_k\left(\left(\nu\beta_k^2-\lambda_k^2\right)\mathrm{ch}(\beta_k x)-\left(\nu\overline{\beta}_k^2-\lambda_k^2\right)\frac{\mathrm{ch}\beta_k^*}{\mathrm{ch}\overline{\beta}_k^*}\mathrm{ch}(\overline{\beta}_k x)\right)+ \quad (21)$$
$$+\sum_{s=1,3,...}^{\infty}(-1)^{\tilde{s}}C_s\mathrm{ch}\xi_s^*\left(\begin{array}{l}\left(\xi_s^2-\nu\mu_s^2\right)-\\ -\dfrac{\xi_s\left[\xi_s^2-(2-\nu)\mu_s^2\right]}{\overline{\xi}_s\left[\overline{\xi}_s^2-(2-\nu)\mu_s^2\right]}\left(\overline{\xi}_s^2-\nu\mu_s^2\right)\mathrm{th}\,\xi_s^*\,\mathrm{cth}\,\overline{\xi}_s^*\end{array}\right)\cos(\mu_s x)=0.$$

System (21) is simplified by expanding the hyperbolic functions into Fourier series with respect to $\cos(\lambda_k y)$ in the first equation (in the second series), and with respect to $\cos(\mu_s x)$ in the second equation (in the first series).

Let us use the well-known expansions [15]:

$$
\begin{aligned}
\operatorname{ch}(\xi_s y) &= \frac{2}{\xi_s}\operatorname{sh}\xi_s^* + 4\xi_s \operatorname{sh}\xi_s^* \sum_{k=1}^{\infty}(-1)^k \frac{\cos(\lambda_k y)}{\xi_s^2+\lambda_k^2},\\
\operatorname{ch}(\overline{\xi}_s y) &= \frac{2}{\overline{\xi}_s}\operatorname{sh}\overline{\xi}_s^* + 4\overline{\xi}_s \operatorname{sh}\overline{\xi}_s^* \sum_{k=1}^{\infty}(-1)^k \frac{\cos(\lambda_k y)}{\overline{\xi}_s^2+\lambda_k^2},\\
\operatorname{ch}(\beta_k x) &= -\frac{4}{\gamma}\operatorname{ch}\beta_k^* \sum_{s=1,3,\ldots}^{\infty}(-1)^{\tilde{s}} \frac{\mu_s\cos(\mu_s x)}{\beta_k^2+\mu_s^2},\\
\operatorname{ch}(\overline{\beta}_k x) &= -\frac{4}{\gamma}\operatorname{ch}\overline{\beta}_k^* \sum_{s=1,3,\ldots}^{\infty}(-1)^{\tilde{s}} \frac{\mu_s\cos(\mu_s x)}{\overline{\beta}_k^2+\mu_s^2}.
\end{aligned}
\tag{22}
$$

The first two expansions in system (22) have free terms, so substituting them into the first equation (21) also produces a free term, which we denote as

$$
G = 2\sum_{s=1,3,\ldots}^{\infty}\frac{\mu_s}{\xi_s}\operatorname{sh}\xi_s^*\left(1-\frac{\xi_s^2\left[\xi_s^2-(2-\nu)\mu_s^2\right]}{\overline{\xi}_s^2\left[\overline{\xi}_s^2-(2-\nu)\mu_s^2\right]}\right)C_s. \tag{23}
$$

To compensate for this part of the rotation angles of the clamped edges, we add an additional deflection to the function (20):

$$
w_0(x) = R_1\operatorname{ch}(\omega x) + R_2\cos(\omega x), \tag{24}
$$

where R_1, R_2, ω are undefined coefficients.

Let us require for function (24) to obey the main equation of problem (11). We then obtain that $\omega = \sqrt{\Omega}$.

We require for boundary conditions to be satisified on the clamped faces, taking into account the free term (23):

$$
\begin{aligned}
R_1\operatorname{ch}\omega_* + R_2\cos\omega_* &= 0,\\
\omega(R_1\operatorname{sh}\omega_* - R_2\sin\omega_*) &= -G.
\end{aligned}
\tag{25}
$$

Then,

$$
R_1 = -\frac{G}{\omega}\frac{1}{\operatorname{ch}\omega_*(\operatorname{th}\omega_* + \operatorname{tg}\omega_*)},\; R_2 = \frac{G}{\omega}\frac{1}{\cos\omega_*(\operatorname{th}\omega_* + \operatorname{tg}\omega_*)}. \tag{26}
$$

where $\omega_* = \omega\gamma/2$.

Now the first equation (21) (without free terms) takes the following form:

$$
\begin{aligned}
&\sum_{k=1}^{\infty}(-1)^k A_k \operatorname{ch}\beta_k^*\left(\beta_k\operatorname{th}\beta_k^* - \overline{\beta}_k\operatorname{th}\overline{\beta}_k^*\right)\cos(\lambda_k y) +\\
&+\sum_{s=1,3,\ldots}^{\infty}\mu_s C_s\left(\begin{aligned}&4\xi_s\operatorname{sh}\xi_s^*\sum_{k=1}^{\infty}(-1)^k\frac{\cos(\lambda_k y)}{\xi_s^2+\lambda_k^2} -\\ &-\frac{\xi_s\left[\xi_s^2-(2-\nu)\mu_s^2\right]\operatorname{sh}\xi_s^*}{\overline{\xi}_s\left[\overline{\xi}_s^2-(2-\nu)\mu_s^2\right]\operatorname{sh}\overline{\xi}_s^*}4\overline{\xi}_s\operatorname{sh}\overline{\xi}_s^*\sum_{k=1}^{\infty}(-1)^k\frac{\cos(\lambda_k y)}{\overline{\xi}_s^2+\lambda_k^2}\end{aligned}\right) = 0.
\end{aligned}
\tag{27}
$$

Rearranging the summation signs with respect to the subscripts k and s and then eliminating the summation sign with respect to the subscript k in the whole expression, we obtain that

$$\operatorname{ch}\beta_k^*\left(\beta_k \operatorname{th}\beta_k^* - \bar{\beta}_k \operatorname{th}\bar{\beta}_k^*\right)A_k + \\ +4\sum_{s=1,3,\ldots}^{\infty}\mu_s\xi_s\left(\frac{1}{\xi_s^2+\lambda_k^2} - \frac{\left[\xi_s^2-(2-\nu)\mu_s^2\right]}{\left[\bar{\xi}_s^2-(2-\nu)\mu_s^2\right]}\frac{1}{\bar{\xi}_s^2+\lambda_k^2}\right)C_s \operatorname{sh}\xi_s^* = 0. \tag{28}$$

Function (24) violates the first boundary condition (8) with respect to the bending moment at the edges $y = \pm 1/2$:

$$M_{y0} = \nu\omega^2(R_1 \operatorname{ch}(\omega x) - R_2\cos(\omega x)). \tag{29}$$

We expand this residual into a Fourier series with respect to $\cos(\mu_s x)$. We use the well-known equations

$$\operatorname{ch}(\omega x) = -\frac{4}{\gamma}\operatorname{ch}\omega_*\sum_{s=1,3,\ldots}^{\infty}\frac{(-1)^{\tilde{s}}\mu_s\cos(\mu_s x)}{\mu_s^2+\omega^2}, \\ \cos(\omega x) = -\frac{4}{\gamma}\cos\omega_*\sum_{s=1,3,\ldots}^{\infty}\frac{(-1)^{\tilde{s}}\mu_s\cos(\mu_s x)}{\mu_s^2-\omega^2} \tag{30}$$

Then

$$M_{y0} = -\frac{4\nu\omega^2}{\gamma}\sum_{s=1,3,\ldots}^{\infty}(-1)^{\tilde{s}}\mu_s\left(R_1\frac{\operatorname{ch}\omega_*}{\mu_s^2+\omega^2} - R_2\frac{\cos\omega_*}{\mu_s^2-\omega^2}\right) = \\ = -G\frac{8\nu}{\gamma}\frac{\omega}{\operatorname{th}\omega_* + \operatorname{tg}\omega_*}\sum_{s=1,3,\ldots}^{\infty}\frac{(-1)^{\tilde{s}}\mu_s^3}{\mu_s^4-\omega^4}\cos(\mu_s x). \tag{31}$$

Now we substitute the last two expansions (22) into the second equation (21) and add the residual (31) from w_0:

$$-G\frac{8\nu}{\gamma}\frac{\omega}{\operatorname{th}\omega_* + \operatorname{tg}\omega_*}\sum_{s=1,3,\ldots}^{\infty}\frac{(-1)^{\tilde{s}}\mu_s^3}{\mu_s^4-\omega^4}\cos(\mu_s x) + \\ +\frac{4}{\gamma}\sum_{k=1}^{\infty}A_k\operatorname{ch}\beta_k^*\left(\begin{array}{l} -\left(\nu\beta_k^2-\lambda_k^2\right)\sum\limits_{s=1,3,\ldots}^{\infty}(-1)^{\tilde{s}}\dfrac{\mu_s\cos(\mu_s x)}{\beta_k^2+\mu_s^2} + \\ +\left(\nu\bar{\beta}_k^2-\lambda_k^2\right)\sum\limits_{s=1,3,\ldots}^{\infty}(-1)^{\tilde{s}}\dfrac{\mu_s\cos(\mu_s x)}{\bar{\beta}_k^2+\mu_s^2}\end{array}\right) + \\ +\sum_{s=1,3,\ldots}^{\infty}(-1)^{\tilde{s}}C_s\operatorname{ch}\xi_s^*\left(\begin{array}{l}\left(\xi_s^2-\nu\mu_s^2\right) - \\ -\dfrac{\xi_s\left[\xi_s^2-(2-\nu)\mu_s^2\right]}{\bar{\xi}_s\left[\bar{\xi}_s^2-(2-\nu)\mu_s^2\right]}\left(\bar{\xi}_s^2-\nu\mu_s^2\right)\operatorname{th}\xi_s^*\operatorname{cth}\bar{\xi}_s^*\end{array}\right)\cos(\mu_s x) = 0. \tag{32}$$

Similarly to the above calculations, we rearrange the summation signs and then eliminate the sign of the external sum. We obtain

$$-G\frac{8\nu}{\gamma}\frac{\omega}{\operatorname{th}\omega_* + \operatorname{tg}\omega_*}\frac{\mu_s^3}{\mu_s^4-\omega^4} + \frac{4\mu_s}{\gamma}\sum_{k=1}^{\infty}A_k\operatorname{ch}\beta_k^*\left(-\frac{\nu\beta_k^2-\lambda_k^2}{\beta_k^2+\mu_s^2} + \frac{\nu\bar{\beta}_k^2-\lambda_k^2}{\bar{\beta}_k^2+\mu_s^2}\right) + \\ +C_s\operatorname{ch}\xi_s^*\left(\left(\xi_s^2-\nu\mu_s^2\right) - \frac{\xi_s\left[\xi_s^2-(2-\nu)\mu_s^2\right]}{\bar{\xi}_s\left[\bar{\xi}_s^2-(2-\nu)\mu_s^2\right]}\left(\bar{\xi}_s^2-\nu\mu_s^2\right)\operatorname{th}\xi_s^*\operatorname{cth}\bar{\xi}_s^*\right) = 0. \tag{33}$$

It follows from expression (28) that

$$A_k^* = -4\frac{\sum\limits_{s=1,3,\ldots}^{\infty} \mu_s \xi_s \left(\frac{1}{\xi_s^2 + \lambda_k^2} - \frac{\left[\xi_s^2 - (2-\nu)\mu_s^2\right]}{\left[\overline{\xi}_s^2 - (2-\nu)\mu_s^2\right]} \frac{1}{\overline{\xi}_s^2 + \lambda_k^2} \right) C_s^*}{\beta_k \operatorname{th}\beta_k^* - \overline{\beta}_k \operatorname{th}\overline{\beta}_k^*}, \tag{34}$$

where we introduce the notations $A_k^* = A_k \operatorname{ch}\beta_k^*$, $C_s^* = C_s \operatorname{sh}\xi_s^*$.

It follows from expression (33) that

$$C_s^* = \frac{G\dfrac{8\nu}{\gamma}\dfrac{\omega}{\operatorname{th}\omega_* + \operatorname{tg}\omega_*}\dfrac{\mu_s^3}{\mu_s^4 - \omega^4} - \dfrac{4\mu_s}{\gamma}\sum\limits_{k=1}^{\infty} A_k^* \left(-\dfrac{\nu\beta_k^2 - \lambda_k^2}{\beta_k^2 + \mu_s^2} + \dfrac{\nu\overline{\beta}_k^2 - \lambda_k^2}{\overline{\beta}_k^2 + \mu_s^2} \right)}{\operatorname{cth}\xi_s^* \left(\left(\xi_s^2 - \nu\mu_s^2\right) - \dfrac{\xi_s\left[\xi_s^2 - (2-\nu)\mu_s^2\right]}{\overline{\xi}_s\left[\overline{\xi}_s^2 - (2-\nu)\mu_s^2\right]} \left(\overline{\xi}_s^2 - \nu\mu_s^2\right) \operatorname{th}\xi_s^* \operatorname{cth}\overline{\xi}_s^* \right)}, \tag{35}$$

and then the expression (23) takes the form

$$G = 2\sum_{s=1,3,\ldots}^{\infty} \frac{\mu_s}{\xi_s} C_s^* \left(1 - \frac{\xi_s^2\left[\xi_s^2 - (2-\nu)\mu_s^2\right]}{\overline{\xi}_s^2\left[\overline{\xi}_s^2 - (2-\nu)\mu_s^2\right]} \right). \tag{36}$$

If we substitute the coefficients A^*_k from Eq. (34) and the quantity G from Eq. (36) into expression (35), then expression (35) becomes an infinite homogeneous system of linear algebraic equations with respect to one sequence of coefficients C^*_s, which contains as a parameter the required natural oscillation frequency Ω. An infinite set of these frequencies can be found from the condition that the determinant of system (35) be equal to zero and the solution of the corresponding frequency equation.

This problem is very complex. To solve it, we propose to use an iterative process of calculating the coefficients C^*_s to determine the natural frequency by searching through this parameter. The coefficients C^*_s in the left-hand side of system (35) can be regarded as the subsequent iteration, and the same coefficients under the sum sign in the right-hand side can be regarded as the previous iteration. For the initial iteration, all coefficients C^*_{s0} can be taken equal to unity or terms of an arbitrary decreasing sequence, for example $C^*_{s0} = 1/\mu^2_s$.

Calculating the right-hand sides of the equations of system (35), we obtain the coefficients C^*_{s1} of the first iteration, which we then substitute into the right-hand sides, and the process repeats. The coefficients are printed at each iteration, and the frequency at which, starting from some iteration, the corresponding coefficients do not differ from each other and are different from zero is the required natural frequency. The numerical implementation of the method assumes that a reduced system (35) is considered, where the number of equations can be varied within a wide range, along with the number of terms in the series whose convergence is analyzed.

The final expression for the deflection function of the plate is written as follows:

$$\begin{aligned} w(x,y) = {} & \frac{G}{\omega}\frac{1}{(\operatorname{th}\omega_* + \operatorname{tg}\omega_*)}\left(-\frac{\operatorname{ch}(\omega x)}{\operatorname{ch}\omega_*} + \frac{\cos(\omega x)}{\cos\omega_*} \right) + \\ & + \sum_{k=1}^{\infty} (-1)^k A_k^* \left(\frac{\operatorname{ch}(\beta_k x)}{\operatorname{ch}\beta_k^*} - \frac{\operatorname{ch}(\overline{\beta}_k x)}{\operatorname{ch}\overline{\beta}_k^*} \right)\cos(\lambda_k y) + \\ & + \sum_{s=1,3,\ldots}^{\infty} (-1)^{\tilde{s}} C_s^* \left(\frac{\operatorname{ch}(\xi_s y)}{\operatorname{sh}\xi_s^*} - \frac{\xi_s\left[\xi_s^2 - (2-\nu)\mu_s^2\right]\operatorname{ch}(\overline{\xi}_s y)}{\overline{\xi}_s\left[\overline{\xi}_s^2 - (2-\nu)\mu_s^2\right]\operatorname{sh}\overline{\xi}_s^*} \right)\cos(\mu_s x). \end{aligned} \tag{37}$$

Numerical results and discussion

To find the natural frequencies and eigenmodes of the CFCF plate in the Maple computing software, we compiled a program in which the ratio of the sides of the plate γ, Poisson's ratio, the number of terms in the series and the number of iterations can be varied to achieve the given calculation accuracy. The main parameter is the vibration frequency, which was searched by the so-called shooting method in combination with the method of successive approximations, until the natural frequency was found. The criterion for the assigned frequency to be accepted as the natural one was for the corresponding coefficients of the series to be immutable starting from a certain iteration, provided that these coefficients are different from zero.

The table shows the first three relative natural frequencies found for symmetric eigenmodes of a square plate (see Note 2 about the third value of Ω_7). Poisson's ratio was taken equal to 0.3. For comparison, experimental data are presented, as well as the corresponding frequencies obtained by other authors using approximate methods. The numbering of natural frequencies in Table (1, 3, 7) corresponds to their ordinal number and takes into account antisymmetric forms (AA) (not given in Table), as well as mixed forms (AS and SA) obtained by different authors.

We should note that the absolute values of the natural vibration frequencies of the plate with specific dimensions and properties appear in the experimental data in [13]. We converted these values to the standard dimensionless form; they are given in Table. Fig. 2 shows the corresponding symmetric 3D eigenmodes.

Searching for natural frequencies, we sequential retained 25, 35 and 50 terms in the series. The number of iterations was 20. The values of the first frequency were varied as follows:

$$22.1178;\ 22.1172;\ 22.1166.$$

A further increase in the terms in the series and the number of iterations did not change the last digit.

The values of the second frequency:

$$43.5407;\ 43.5410;\ 43.5410.$$

The values of the third frequency:

$$87.4371;\ 87.4371;\ 87.4370.$$

The presented data indicate that the natural frequencies were found with high accuracy.

Table

Obtained natural frequencies of symmetric vibrations in a square CFCF plate: comparison with the literature data

Relative natural frequency		Authors
Ω_1	Ω_3	
Experiment		
22.206	42.840	Singhal et al. [13]
Computational results		
22.1166	43.5410	This paper
22.272	43.664	Leissa [2]
22.03	43.20	Mizusawa [9]
22.22	43.91	Lew et al. [8]
22.17	43.60	Kshirsagar et al. [10]
22.168	43.596	Monterubbio et al. [3]

Notes. 1. $\Omega = p\sqrt{\rho h b^4/D}$, where p is the required vibration frequency of the plate; ρ is the density of its material; h, b are the thickness and side of the square plate; D is its cylindrical stiffness. 2. According to our data, Ω_7 = 87.4370 (obtained for the first time).

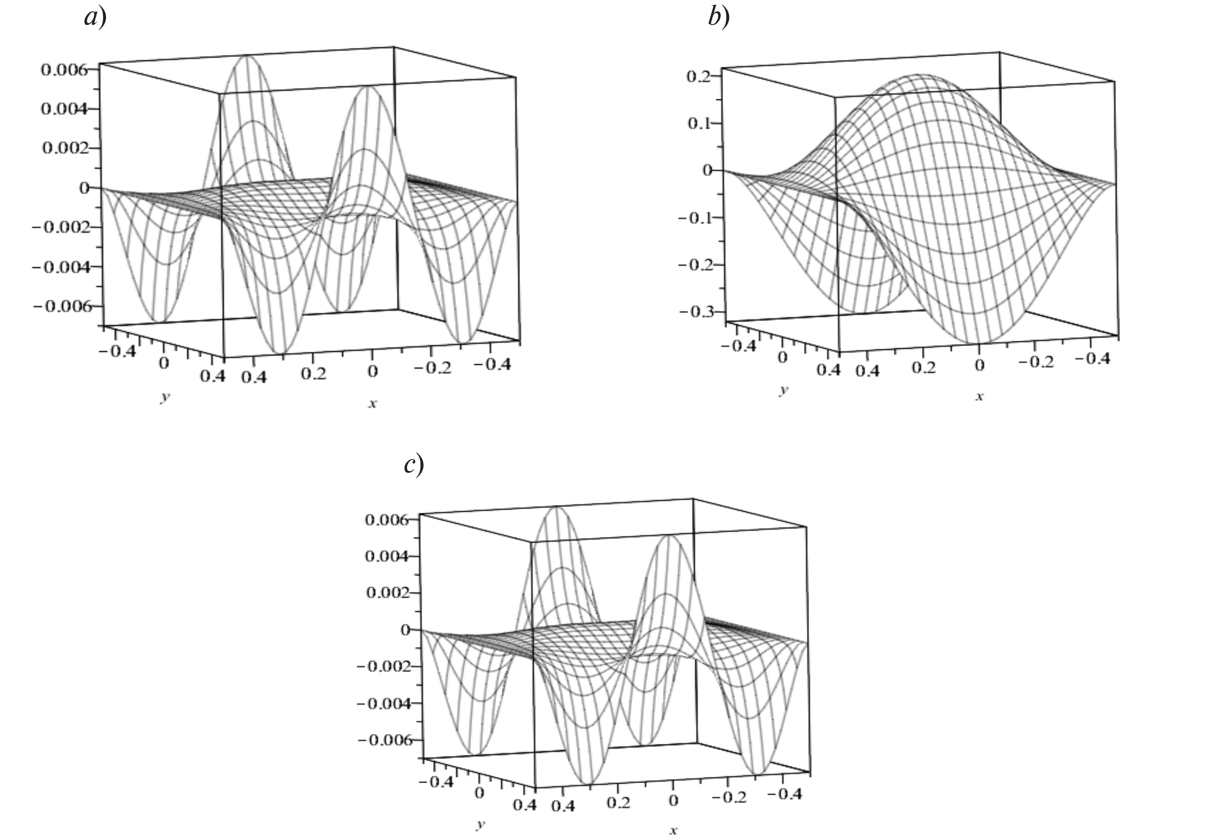

Fig. 2. First (*a*), second (*b*) and third (*c*) SS-eigenmodes of the CFCF plate
(the values of Ω_1, Ω_3, Ω_7 are given in Table)

Comparison with experimental data [13] (for two frequencies) shows that the results are very close, but the value of the second frequency in [13] is somewhat lower, evidently because it was impossible to provide strictly rigid clamping of two parallel sides of the plate in the experiment.

Approximate calculations by variational methods (for example, the Ritz or Kantorovich methods) generally produce overestimated values of the required parameters, which is actually observed in this case (see the results in [2, 3, 8]). The exception is [9], where the frequencies are lower than in our paper. This is most likely the result of a lower frequency estimate, which stems from using B-splines.

Accounting for the initial conditions of the problem

The problem of free vibrations of a rectangular CFCF panel will be solved to completion if arbitrary constants c_1 and c_2 of the time function (9) from the initial conditions (1) are found. Since the eigenvalue problem has an infinite set of roots p_i, the initial deflection function (6) takes the form

$$W(x,y,t)=\sum_{i=1}^{\infty}\left(c_{1i}\cos p_i t+c_{2i}\sin p_i t\right)w_i(x,y), \tag{38}$$

where $w_i(x,y)$ are the eigenfunctions of the problem of the form (37) for eigenvalues p_i.

Notice that the initial conditions cannot be set arbitrarily. First of all, the initial deflections $W_0(X, Y)$ and the initial velocities $V_0(X, Y)$ must be equal to zero at the clamped edges of the plate. Defining (or setting) initial conditions is often a separate problem.

The simplest case is when the initial shape of the plate is the shape of its curved surface under the action of a uniform transverse load. After such a load is instantly removed, the plate begins to make free vibrations. These initial conditions often occur during operation of the structure.

However, this implies that the problem of bending in a plate under the action of this stationary load has been solved. If we consider a freely supported plate or a plate with two supported parallel faces and two arbitrarily fixed faces, the solution to this problem is obtained in a closed form, and it can be used.

There is no such closed solution for a CFCF plate. Then we can use one of the known approximate solutions to the corresponding problem of bending in a CFCF plate; we denote it as $w_{st}(x,y)$.

If the initial velocities of the points of the plate are zero, then all arbitrary constants $c_{2i} = 0$ and we can write the equality

$$\sum_{i=1}^{\infty} c_{1i} w_i(x, y) = w_{st}(x, y). \tag{39}$$

To find the coefficients c_{1i}, we need to represent the coordinate functions in the left and right-hand sides of the equation by double Fourier series, which gives an infinite system of linear algebraic equations with respect to these coefficients.

Thus, the problem of free transverse vibrations of a rectangular plate can be solved to completion. The solution algorithm is similar for other types of initial conditions.

Conclusion

We constructed an effective algorithm for finding the natural frequencies of symmetric eigenmodess of a rectangular plate with two clamped opposite and two free parallel edges. Numerical values of epy frequencies are obtained with high accuracy, and the corresponding 3D eigenmodes are given.

Our findings can be useful for design organizations to calculate the resonance of plane elements in various structures.

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