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Effects observed in the ballistic-conductive model of heat conduction

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Abstract. In this paper, we study the behavior of solutions to the initial value problem in the framework of the ballistic-conductive (BC) model of heat conduction. As a result of the study, the effect of partial "immobilization" of thermal energy has been found. This effect is unphysical and is a defect of the BC model.

Keywords: non-Fourier heat conduction, hyperbolic heat conduction, the ballistic-conductive model

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Эффект в баллистико-кондуктивной модели теплопроводности

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Аннотация. В этой статье мы рассматриваем поведение решений задачи Коши в рамках баллистико-кондуктивной (БК) модели теплопроводности. В результате исследования обнаружен эффект частичной «иммобилизации» тепловой энергии. Этот эффект нефизичен и является дефектом БК модели.

Ключевые слова: неклассическая теплопроводность, гиперболическая теплопроводность, баллистико-кондуктивная модель

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Introduction

The heat equation, based on Fourier's law, is commonly used for description of thermal conductivity. However, Fourier's law is valid under the assumption of local equilibrium [1], which is violated in very small dimensions and short timescales, and at low temperatures [2–6]. There are a number of methods for constructing models of non-Fourier thermal conductivity [7–16].

In Ref. [15], a ballistic-conductive (BC) model of heat conduction in the framework of non-equilibrium thermodynamics with internal variables (NET-IV) was developed. The BC model is described by a symmetric hyperbolic system. This provides a finite velocity of thermal energy propagation. However, the finiteness of the propagation velocity is not enough for the model to accurately describe heat transfer. In Ref. [17] this model was tested against experimental data and demonstrated qualitative agreement.

In Refs. [18, 19], we investigated the dual-phase-lag model of heat conduction [4] and showed unphysical effects of this model. In this paper, in order to find out the features of the BC model and to establish its possible limitations, we study the behavior of solutions to the initial value problem in the framework of the BC model. Asymptotic analysis allows us to establish the structure and properties of the solutions without the influence of boundary conditions.

Statement of the problem

Consider the system of dimensionless equations, describing the ballistic–conductive (BC) model of heat conduction [15, 17]:

$$\begin{gathered}\partial_t T + \partial_x q = f(x,t),\\ \tau_q \partial_t q + q + \partial_x T + k\partial_x Q = 0,\\ \tau_Q \partial_t Q + Q + k\partial_x q = 0,\end{gathered} \tag{1}$$

or, equivalently, $\partial_t \boldsymbol{w} + A\partial_x \boldsymbol{w} + B\boldsymbol{w} = (f,0,0)$, where $\boldsymbol{w} = (T,q,Q)$, T is temperature, q is heat flux, Q is an internal variable, $f \equiv f(x, t)$ is a heat source, τ_q and τ_Q are relaxation times, and k is a material parameter. The system (1) is symmetric hyperbolic [20], and the eigenvalues of the matrix A are $\lambda_1 = 0$ and $\lambda_{2,3} = \pm v$, $v = [(k^2 + \tau_Q)/\tau_q\tau_Q]^{1/2}$.

Excluding in the system (1) the variables q and Q, we find that temperature satisfies the equation

$$\begin{gathered}\tau_q\tau_Q\partial_t^3 T + \left(\tau_q + \tau_Q\right)\partial_t^2 T + \partial_t T - \partial_x^2 T - \left(k^2 + \tau_Q\right)\partial_t\partial_x^2 T = \\ = \tau_q\tau_Q\partial_t^2 f + \left(\tau_q + \tau_Q\right)\partial_t f + f - k^2\partial_x^2 f,\end{gathered} \tag{2}$$

and we consider the equation on the whole real axis. Note that, if $f = 0$, Eq. (2) takes the form of Eq. (14) in Ref. [17]. We impose the following initial conditions on the system (1): $T|_{t=0} = \varphi(x)$, $q|_{t=0} = 0$, $Q|_{t=0} = 0$. From these conditions and the system (1) we obtain initial conditions for Eq. (2):

$$T\Big|_{t=0} = \varphi(x),\ \ \partial_t T\Big|_{t=0} = f\Big|_{t=0},\ \ \partial_t^2 T\Big|_{t=0} = \tau_q^{-1}\varphi''(x) + \partial_t f\Big|_{t=0}. \tag{3}$$

Solution to the initial value problem

In this section we assume that $f = 0$, since below we show that the solution to the initial value problem (2), (3) is the same in the cases $\varphi = \varphi_0(x)$, $f = 0$ and $\varphi = 0$, $f = \varphi_0(x)\delta(t)$, where $\delta(\cdot)$ is the Dirac delta function. If $f = 0$, Eq. (2) takes the form

$$\tau_q\tau_Q\partial_t^3 T + \left(\tau_q + \tau_Q\right)\partial_t^2 T + \partial_t T - \partial_x^2 T - \left(k^2 + \tau_Q\right)\partial_t\partial_x^2 T = 0, \tag{4}$$

and the initial conditions (3) take the form

$$T\big|_{t=0} = \varphi(x),\ \ \partial_t T\big|_{t=0} = 0,\ \ \partial_t^2 T\big|_{t=0} = \tau_q^{-1}\varphi''(x). \tag{5}$$

The Fourier transform of the temperature distribution is given by

$$\mathcal{F}T = \left\{ \mathrm{e}^{-\mu_1 t} E + \mathrm{e}^{-\mu_2 t} \left[F\cos Bt + G\frac{\sin Bt}{B} \right] \right\} \mathcal{F}\varphi, \tag{6}$$

(see the derivation in the next section), the coefficients are given by Eqs. (13), (14), (15).

Suppose that $\varphi(x) = \delta(x)$, i.e., all the thermal energy at the initial moment was concentrated at the origin. In this case the Fourier transform of the temperature distribution, which we denote by T_δ, is given by

$$\mathcal{F}T_\delta = \mathrm{e}^{-\mu_1 t} E + \mathrm{e}^{-\mu_2 t} \left[F\cos Bt + G\frac{\sin Bt}{B} \right].$$

Taking into account the asymptotic behavior of the coefficients, we obtain the asymptotic behavior of the Fourier transform

$$\mathcal{F}T_\delta = \mathrm{e}^{-\mu_{1,\infty} t} E_\infty + \mathrm{e}^{-\mu_{2,\infty} t} \left[F_\infty \cos v\xi t + G_\infty \frac{\sin v\xi t}{v\xi} \right] + O\left(\frac{1}{\xi^2}\right) \quad \text{as} \quad \xi \to \infty.$$

Performing the inverse Fourier transform we conclude that the temperature distribution has the form $T_\delta(x,t) = T_{\mathrm{sing},1}(x,t) + T_{\mathrm{sing},2}(x,t) + T_{\mathrm{disc}}(x,t) + T_{\mathrm{cont}}(x,t)$, where $T_{\mathrm{sing},1}(x,t) = \mathrm{e}^{-\mu 1,\infty t} E_\infty \delta(x)$ and $T_{\mathrm{sing},2}(x,t) = \mathrm{e}^{-\mu 2,\infty t} F_\infty [\delta\,(x - vt) + \delta\,(x + vt)]/2$ are singular terms, $T_{\mathrm{disc}}(x,t) = \mathrm{e}^{-\mu 2,\infty t} G_\infty \mathbf{1}_{(-vt,vt)}/2v$ is a discontinuous term. The term T_{cont} is a continuous function, since its Fourier transform has the asymptotic behavior $O(1/\xi^2)$ as $\xi \to \infty$.

In the general case the temperature distribution is the convolution of T_δ with the initial temperature distribution φ, i.e.,

$$T(x,t) = \int_{-\infty}^{\infty} T_\delta\,(x-y,t)\varphi(y)\mathrm{d}x \equiv T_1\,(x,t) + T_2\,(x,t) + T_3\,(x,t), \tag{7}$$

where

$$T_1\,(x,t) = \int_{-\infty}^{\infty} T_{\mathrm{sing},1}(x-y,t)\varphi(y)\mathrm{d}x \equiv \mathrm{e}^{-\mu_{1,\infty} t} E_\infty \varphi(x), \tag{8}$$

$$T_2\,(x,t) = \int_{-\infty}^{\infty} T_{\mathrm{sing},2}(x-y,t)\varphi(y)\mathrm{d}x \equiv \mathrm{e}^{-\mu_{2,\infty} t} F_\infty \frac{1}{2}\big[\varphi(x-vt) + \varphi(x+vt)\big], \tag{9}$$

and T_3 is the convolution of $T_{\mathrm{disc}} + T_{\mathrm{cont}}$ with φ. The term T_1 means that part of the initial thermal energy does not spread anywhere, though this part decreases exponentially with time. This is an unphysical effect of the BC model. The mathematical reason for this effect is the zero eigenvalue of the matrix A of the system (1). The term T_2 means that equal parts of the initial thermal energy propagate in opposite directions with the velocity v, while exponentially decaying. The term T_2 corresponds to ballistic phonons.

Fig. 1 presents solution (7) to problem (4), (5). The values of the parameters τ_q, τ_Q and k have been taken from Ref. [21]. We take the function $\varphi_\sigma(x) = [(2\pi)^{1/2}\sigma]^{-1} \exp(-x^2/2\sigma^2)$ as an initial temperature distribution. The figure clearly shows the unphysical term (8) and the “ballistic” term (9). The figure also shows that the portion of thermal energy contained in the unphysical term is comparable to that in the “ballistic” term.

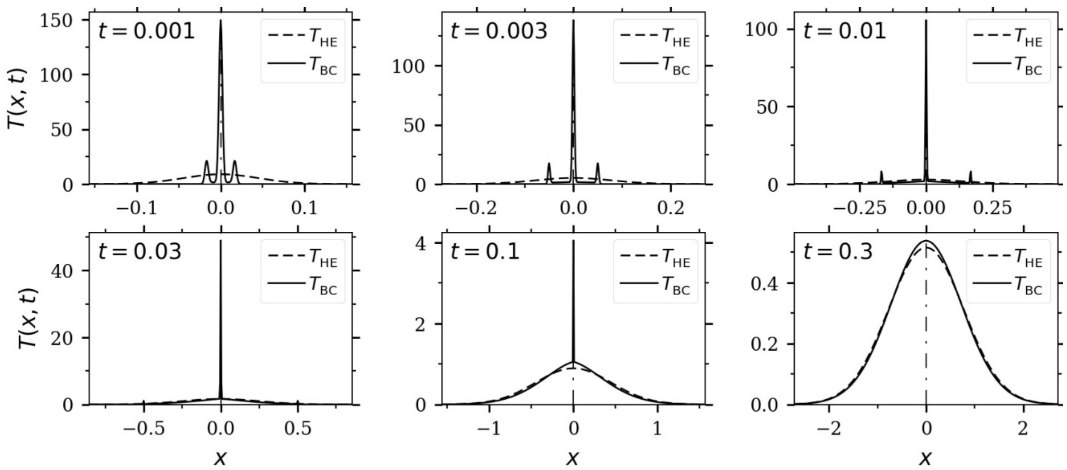

Fig. 1. Solution T_{BC} to the initial value problem in the framework of the ballistic-conductive model in comparison with the solution T_{HE} to the initial value problem for the heat equation. The parameters are τ_q =0.0156, τ_Q =0.0058 and k =0.140, σ =0.002

The solution (7) is compared in the figures with the solution to the initial value problem for the heat equation

$$\partial_t T - \partial_x^2 T = 0, \;\; T\big|_{t=0} = \varphi(x).$$

Derivation of the solution to the initial value problem

The Fourier transform of the problem (2), (3) results in the equation

$$\begin{gathered}\tau_q\tau_Q\partial_t^3\mathcal{F}T + \left(\tau_q+\tau_Q\right)\partial_t^2\mathcal{F}T + \partial_t\mathcal{F}T + \xi^2\mathcal{F}T + \left(k^2+\tau_Q\right)\xi^2\partial_t\mathcal{F}T = \\ = \tau_q\tau_Q\partial_t^2\mathcal{F}f + \left(\tau_q+\tau_Q\right)\partial_t\mathcal{F}f + \mathcal{F}f + k^2\xi^2\mathcal{F}f\end{gathered} \tag{10}$$

with initial conditions

$$\mathcal{F}T\big|_{t=0} = \mathcal{F}\varphi, \;\; \partial_t\mathcal{F}T\big|_{t=0} = \mathcal{F}f\big|_{t=0}, \;\; \partial_t^2 T\big|_{t=0} = -\xi^2\tau_q^{-1}\mathcal{F}\varphi + \partial_t\mathcal{F}f\big|_{t=0}. \tag{11}$$

The Laplace transform of the problem (10), (11) results in the equation

$$\begin{gathered}\tau_q\tau_Q\left(s^3\mathcal{LF}T - s^2\mathcal{F}\varphi + \frac{\xi^2}{\tau_q}\mathcal{F}\varphi\right) + \left(\tau_q+\tau_Q\right)\left(s^2\mathcal{LF}T - s\mathcal{F}\varphi\right) + \left(s\mathcal{LF}T - \mathcal{F}\varphi\right) + \\ + \xi^2\mathcal{LF}T + \left(k^2+\tau_Q\right)\xi^2\left(s\mathcal{LF}T - \mathcal{F}\varphi\right) = \left[\tau_q\tau_Q s^2 + \left(\tau_q+\tau_Q\right)s + 1 + k^2\xi^2\right]\mathcal{LF}f.\end{gathered}$$

Hence, the Laplace–Fourier transform of the temperature distribution is given by

$$\mathcal{LF}T = \frac{\tau_q\tau_Q s^2 + \left(\tau_q+\tau_Q\right)s + 1 + k^2\xi^2}{\tau_q\tau_Q s^3 + \left(\tau_q+\tau_Q\right)s^2 + \left[1+\left(k^2+\tau_Q\right)\xi^2\right]s + \xi^2}\left(\mathcal{F}\varphi + \mathcal{LF}f\right). \tag{12}$$

One can see that the solution is the same in the cases $\varphi = \varphi_0(x)$, $f = 0$ and $\varphi = 0$, $f = \varphi_0(x)\,\delta(t)$, where $\delta(\cdot)$ is the Dirac delta function. Therefore, hereafter we assume that $f = 0$. If $f = 0$, Eq. (12) takes the form

$$\mathcal{L}\mathcal{F}T = \frac{s^2 + as + (\tau_q\tau_Q)^{-1}\left(1 + k^2\xi^2\right)}{s^3 + as^2 + bs + c}\mathcal{F}\varphi = \left[\frac{E}{u - 2A} + \frac{F(u + A) + G}{\left(u + A\right)^2 + B^2}\right]\mathcal{F}\varphi, \tag{13}$$

where

$$a = \tau_q^{-1} + \tau_Q^{-1},\ b = (\tau_q\tau_Q)^{-1} + v^2\xi^2,\ c = (\tau_q\tau_Q)^{-1}\xi^2,$$
$$s = u - a/3,$$
$$p \equiv p(\xi) = -(a^2/3) + b,\ q \equiv q(\xi) = 2\left(a/3\right)^3 - (ab/3) + c,$$
$$\alpha = \sqrt[3]{-(q/2) + \sqrt{\Delta}},\ \beta = \sqrt[3]{-(q/2) - \sqrt{\Delta}},\ \Delta = \left(p/3\right)^3 + \left(q/2\right)^2,$$
$$A = (\alpha + \beta)/2,\ B = \sqrt{3}(\alpha - \beta)/2,\ C = a/3,\ D = -2\left(a/3\right)^2 + (\tau_q\tau_Q)^{-1}(1 + k^2\xi^2),$$

the roots α and β are chosen so that the equality $\alpha\beta = -p/3$ is valid and the value A is real, and

$$E = \frac{4A^2 + 2AC + D}{9A^2 + B^2},\ F = 1 - E,\ G = \frac{-3A^3 + AB^2 + 3A^2C + B^2C - 3AD}{9A^2 + B^2}. \tag{14}$$

We can conclude that the inverse Laplace transform is equal to

$$\mathcal{L}^{-1}\left[\frac{E}{u - 2A} + \frac{F(u + A) + G}{\left(u + A\right)^2 + B^2}\right] = \mathrm{e}^{-\mu_1 t}E + \mathrm{e}^{-\mu_2 t}\left[F\cos Bt + G\frac{\sin Bt}{B}\right],$$

where

$$\mu_1 \equiv \mu_1(\xi) = -2A + \left(\tau_q^{-1} + \tau_Q^{-1}\right)/3,\ \mu_2 \equiv \mu_2(\xi) = A + \left(\tau_q^{-1} + \tau_Q^{-1}\right)/3. \tag{15}$$

As a result, we obtain that the Fourier transform of the temperature distribution is given by Eq. (6).

Asymptotic behavior of the coefficients as $\xi \to \infty$

The asymptotic behavior of the coefficients p and q is given by

$$p = C_p\xi^2\left[1 + O\left(\xi^{-2}\right)\right] \text{ and } q = C_q\xi^2\left[1 + O\left(\xi^{-2}\right)\right],$$

where

$$C_p = v^2,\ C_q = (\tau_q\tau_Q)^{-1} - \left(\tau_q^{-1} + \tau_Q^{-1}\right)v^2/3.$$

Therefore,

$$\sqrt{\Delta} = \left(C_p/3\right)^{3/2}|\xi|^3\left[1 + O\left(\xi^{-2}\right)\right],$$

and the asymptotic behavior of the radicals α and β is given by

$$\alpha = \left(C_p/3\right)^{1/2}|\xi|\left[1 - \left(C_p/3\right)^{-3/2}(C_q/6)|\xi|^{-1} + O\left(\xi^{-2}\right)\right],$$

$$\beta = -\left(C_p/3\right)^{1/2}|\xi|\left[1 + \left(C_p/3\right)^{-3/2}(C_q/6)|\xi|^{-1} + O\left(\xi^{-2}\right)\right].$$

The asymptotic behavior of the coefficients A, B and D follows from the above asymptotics and is given by

$$A = (\tau_q^{-1} + \tau_Q^{-1})/6 - (k^2 + \tau_Q)^{-1}/2 + O\left(|\xi|^{-1}\right),\ B = \nu|\xi|\left[1 + O\left(\xi^{-2}\right)\right]$$

and

$$\text{and } D = k^2\tau_q^{-1}\tau_Q^{-1}\xi^2\left[1 + O\left(\xi^{-2}\right)\right].$$

Therefore, the asymptotic behavior of the coefficients E, F and G is given by

$$E = E_\infty\left[1 + O\left(\xi^{-2}\right)\right],\ F = F_\infty\left[1 + O\left(\xi^{-2}\right)\right] \text{ and } G = G_\infty\left[1 + O\left(|\xi|^{-1}\right)\right],$$

where

$$E_\infty = \frac{k^2}{k^2 + \tau_Q},\ F_\infty = 1 - E_\infty,\ G_\infty = \frac{2k^2 - \tau_Q}{2(k^2 + \tau_Q)^2}.$$

The asymptotic behavior of the coefficients μ_1 and μ_2 is given by

$$\mu_1 = \mu_{1,\infty} + O\left(\xi^{-2}\right),\ \mu_2 = \mu_{2,\infty} + O\left(\xi^{-2}\right).$$

where

$$\mu_{1,\infty} = (k^2 + \tau_Q)^{-1},\ \mu_{2,\infty} = \left[\tau_q^{-1} + \tau_Q^{-1} - (k^2 + \tau_Q)^{-1}\right]/2.$$

Conclusions

As a result of the study, the effect of partial “immobilization” of thermal energy was found, when part of the initial thermal energy does not spread anywhere, though this part decreases exponentially with time. This is the defect of the BC model. This effect is similar to that previously established for particles in the mass transfer model, which is described by the Jeffreys-type equation [22].

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