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One-dimensional Fokker-Planck equation with relativistic effects for numerical simulations

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Abstract. This paper presents considerations on the topic of creating one-dimensional Fokker-Planck equation with relativistic effects. The derivation of two-dimensional relativistic equation and an attempt to average to the one-dimensional equation are demonstrated. The results are used for numerical simulations of LHCD.

Keywords: tokamak, current drive, helicon, electron distribution function, relativistic Fokker-Planck equation

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Одномерное уравнение Фоккера-Планка с релятивистскими эффектами для численного моделирования

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Аннотация. В данной работе представлены размышления на тему создания одномерного релятивистского уравнения Фоккера–Планка. Демонстрируется вывод двумерного релятивистского уравнения и попытка усреднения к одномерному уравнению. Результаты работы используются для численного моделирования LHCD.

Ключевые слова: токамак, ток увлечения, геликон, функция распределения электронов, релятивистское уравнение Фоккера–Планка

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Introduction

The neutral beam injection and the injection of the electromagnetic waves are main methods of current generation in a tokamak that have been experimentally verified and validated [1]. At present, the method of generating current using slowed-down high-frequency waves of the lower hybrid (LH) frequency range ($\Delta f \approx (1-10)$ GHz) is widely used in classical tokamaks (with an aspect ratio of $R/a > 2$). It has the highest theoretically and experimentally confirmed efficiency. The method is based on the effect of the transmission of a pulse by a slowed-down RF wave in the lower hybrid frequency range to electrons due to Landau damping. As a result, the electron distribution function (EDF) is deformed, which ensures an increase in the total current in the tokamak plasma.

In the FRTC [2, 3] code, the calculation of the quasilinear diffusion coefficient is performed using the ray tracing method, where the wave equations for LH waves are solved under the geometric optics approximation. The injected wave power is absorbed via the Landau resonance by the electrons with velocities equal to the wave's phase velocity. As a result, the process of quasilinear diffusion on waves forms a plateau in the electron distribution function in the region of resonant velocities (there is a transition of particles from the region of lower to higher velocities with a concomitant increase in the kinetic energy of the particles). During the process there are statistically more fast particles than in an equilibrium state, hence a current arises, the time dynamics of which generate a vortex electric field. In its turn, this field begins to accelerate continuously those electrons in the tail of the distribution function, in which the electric field driving force is stronger than the minimum frictional drag force. Thus, a "tail" of fast particles is formed. The problem with the existing method is that this tail can extend up to speeds of 0.5–0.6 c since the code solves the one-dimensional Fokker-Planck equation without relativistic effects. This leads to an increase in the value of the generated current. This can make a negative impact on the planning of further experiments and on the scaling up of this technology to larger machines. Therefore, this paper describes an attempt to develop a one-dimensional relativistic equation, which would be possible to use in conjunction with ray tracing.

Fokker-Planck equation with relativistic effect

Let us consider a wave packet propagating at an angle to the external magnetic field $\boldsymbol{B}_0 = B_{0\boldsymbol{ez}}$ in a homogeneous plasma. The wave field can be expressed as

$$\vec{E}(\vec{r}) = \int_{-\infty}^{+\infty} dk_z \vec{A}(k_z) \exp(ik_x x + ik_z z - i\omega t) + c.c. \tag{1}$$

Vlasov's kinetic equation with an external magnetic field for magnetized plasma is [4]:

$$\left(\frac{\partial}{\partial t} + v_i \frac{\partial}{\partial x_i} + \frac{\omega_c}{\gamma} \frac{\partial}{\partial \theta} - |e| \left(E_i + e_{ijk} v_j B_k \right) \frac{\partial}{\partial p_i} \right) f_e = St(f_e) \tag{2}$$

where θ – azimuth angle of the cylindrical coordinate system in momentum space, $\omega_c = eB_0/mc$, $p_k = \gamma(v) m_e v_k$, $\gamma = \dfrac{1}{\sqrt{1 - \dfrac{v^2}{\tilde{n}^2}}}$ and $St(f_e)$ – collision operator in Landau form.

We will only examine the left-hand side of our equation. We are looking for a solution in the form of $f_e = f_0 + f^{(1)}$, where $f_0(p,t)$ – is an isotropic distribution function and $f^{(1)} = \dfrac{1}{2\pi} \int_{-\infty}^{\infty} dk_z f^{(1)}(k_z)$ – is the correction associated w ith the perturbation by waves in the plasma. Let us denote by v_z, k_z the components of the vectors $\vec{v}$, $\vec{k}$ along $\vec{B}_0$ field, and by $v_\perp$, $k_\perp$ the components in the perpendicular $\vec{B}_0$ plane. Let θ be the angle between $\boldsymbol{v}_\perp$ and the plane $\vec{k}_\perp$, $\vec{B}_0$ (so $[\vec{e}_\perp \times \vec{e}_\theta] = \vec{e}_z$. Substituting all this into equation (1), we can obtain

$$\frac{\partial f}{\partial \vec{p}}[\vec{v}\times\vec{B}_0]=0 \tag{3}$$

$$\left(-i\alpha+i\lambda\cos(\theta)\right)f^{(1)}\left(k_z\right)-\frac{|e|\gamma}{2\omega_c}\left(\vec{A}+\frac{\vec{v}\times\left(\vec{k}\times\vec{A}\right)}{\omega}\right)\frac{\partial f_0}{\partial \vec{p}}=0 \tag{4}$$

$$\alpha=\frac{\left(\omega-k_z\nu_z\right)\gamma}{\omega_c},\ \lambda=\frac{k_x\nu_\perp\gamma}{\omega_c},\ \omega_c=\left|\omega_{ce}\right|.$$

From equation (3) it is clear that $\partial f_0/\partial\theta = 0$, i. e. f_0 can be any function depending only on p_z and $p_\perp$: $f_0 = f_0\ (p_\perp,\ p_\parallel)$. By integrating equation (4), we obtain

$$f^{(1)}(k_z)=\frac{|e|\gamma}{2\omega_c}\exp(i\alpha\theta-i\lambda\sin(\theta))\int_{-\infty}^{\theta}d\theta'\exp(i\lambda\sin(\theta')-i\alpha\theta')\times$$

$$\left(\begin{array}{l} A_x((1-\frac{k_z v_z}{\omega})\cos(\theta')\frac{\partial}{\partial p_\perp}+\frac{k_z v_\perp}{\omega}\cos(\theta')\frac{\partial}{\partial p_z})+ \\ +A_y((1-\frac{k_z v_z}{\omega})\sin(\theta')\frac{\partial}{\partial p_\perp}+\frac{k_z v_\perp}{\omega}\sin(\theta')\frac{\partial}{\partial p_z})+ \\ +A_z((1-\frac{k_x v_\perp}{\omega}\cos(\theta'))\frac{\partial}{\partial p_z}+\frac{k_x v_z}{\omega}\cos(\theta')\frac{\partial}{\partial p_\perp}) \end{array}\right)f_0(p_\perp,p_z). \tag{5}$$

Using the relation $\exp(i\lambda\sin(\theta)=\sum_p J_p(\lambda)\exp(ip\theta)$ and holding back only the terms responsible for Landau's resonant interaction with electrons, we find the correction $f^{(1)}(k_z)$:

$$f^{(1)}\left(k_z\right)=i\frac{|e|\gamma}{2\omega_c}\frac{\exp\left(-i\lambda\sin(\theta)\right)}{\alpha}\vec{l}\left(k_z\right)\cdot\vec{A}\left(k_z\right)\frac{\partial}{\partial p_z}f_0\left(p_\perp,p_\parallel\right) \tag{6}$$

where $\vec{l}(k_z)=\left[0,iJ_1(\lambda)\frac{k_z\nu_\perp}{\omega},J_0(\lambda)\right]_{\omega=k_z\nu_z}$.

Substituting this into (2), we obtain an equation to describe the evolution of the function f_0:

$$\left(\frac{\partial}{\partial t_0}+v_i\frac{\partial}{\partial x_i}+\frac{\omega_c}{\gamma}\frac{\partial}{\partial\theta}\right)f_0-\frac{|e|}{2}\int_{-\infty}^{+\infty}\frac{dk_z}{2\pi}\left(\overline{A^*}+\frac{\vec{v}\times\left(\vec{k}\times\overline{A^*}\right)}{\omega}\right)\frac{\partial f^{(1)}\left(k_z\right)}{\partial\vec{p}}-c.c.=St\left(f_0\right)$$

Again, considering only the resonance terms $\omega = k_z v_z$, after averaging over the angle θ, and averaging over the random phase $\langle(\boldsymbol{l}\cdot\boldsymbol{A}(k_z))^*\ (\boldsymbol{l}\cdot\boldsymbol{A}(k_z'))\rangle = 2\pi|(\boldsymbol{l}\cdot\boldsymbol{A}(k_z))|^2$ we get

$$\frac{\partial f_0}{\partial t}-\frac{i|e|^2}{4}\int_{-\infty}^{\infty}dk_z\frac{\partial}{\partial p_z}\frac{|\vec{l}\left(k_z\right)\cdot\vec{A}\left(k_z\right)|^2}{\omega-k_z v_z}\frac{\partial}{\partial p_z}f_0-c.c.=St\left(f_0\right) \tag{8}$$

Based on the Sokhotski-Plemelj theorem $\int_{-\infty}^{\infty}\frac{f(x)}{x\pm i\varepsilon}dx=\upsilon.p.\int_{-\infty}^{\infty}\frac{f(x)}{x}dx\pm i\pi f(0).$

Reducing the corresponding integrals in the sense of the Cauchy principal value, we get:

$$\frac{\partial f_0}{\partial t}-\frac{\pi|e|^2}{2}\int_{-\infty}^{\infty}dk_z\frac{\partial}{\partial p_z}|\vec{l}\left(k_z\right)\cdot\vec{A}\left(k_z\right)|^2\,\delta\left(\omega-k_z v_z\right)\frac{\partial}{\partial p_z}f_0=St\left(f_0\right) \tag{9}$$

This is the Fokker-Planck equation, where $\frac{\pi|e|^2}{2}\int_{-\infty}^{\infty} dk_z \left|(\boldsymbol{l}\cdot\boldsymbol{A}(k_z))\right|^2 \delta(\omega - k_z v_z)$ – is the diffusion coefficient.

To reduce this equation further to a one-dimensional one it is necessary to make some assumptions on the distribution function. In the absence of collisions, the relativistic collision operator from [5] is going to zero and the solution of the equation should be the Maxwell–Jüttner distribution function [5, 6]:

$$f(\vec{p}) = \frac{1}{4\pi m^3 c^3 \beta K_2\left(\frac{1}{\beta}\right)} \exp\left(-\frac{1}{\beta}\sqrt{1+\left(\frac{p_\perp}{mc}\right)^2+\left(\frac{p_\parallel}{mc}\right)^2}\right) \qquad (10)$$

where $\beta = T/mc^2$ and K2 – is the Macdonald function.

Provided that $p_\perp \ll p_\parallel$

$$f(\vec{p}) \propto \exp\left(-\frac{1}{\beta}\sqrt{1+\left(\frac{p_\perp}{mc}\right)^2+\left(\frac{p_\parallel}{mc}\right)^2}\right) \approx \exp\left(-\frac{1}{\beta}\sqrt{1+\left(\frac{p_\parallel}{mc}\right)^2}\right)\exp\left(-\frac{1}{2\beta}\frac{p_\perp{}^2}{\sqrt{1+\left(\frac{p_\parallel}{mc}\right)^2}}\right) \qquad (11)$$

Thus, the attempt to factorize the equilibrium distribution function leads to a nonlinear dependence of the transverse component on the longitudinal momentum. That is, the transverse temperature in such a case is a function of the longitudinal momentum even in the equilibrium case. Accordingly, in the nonequilibrium case, after exposure to the wave field and vortex field, followed by scattering on the cloud of "warm" particles, the transverse component of the distribution function will continue to depend on $\gamma(p_z)$, but only by a law significantly different from the equilibrium one. Hence, it is necessary to average over an arbitrary transverse distribution function. But then we will obtain moments of distribution function that depend on $\gamma(p_z)$. The calculation of the moments would already require further assumptions on the form of the transverse component $f(p)$. Any such assumptions introduce only additional inaccuracy into the calculations, compared with the one-dimensional model without relativism.

Conclusion

As a result, the correct approach from the computational point of view is the subsequent refinement of the two-dimensional model to calculate the distribution function using the ray-tracing module. This approach will allow us to calculate the scattering of particles more correctly and, among other things, to take into account such effect as backward runaway electrons.

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