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Application of global stability analysis to predicting characteristics of Tollmien-Schlichting waves

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Abstract. A numerical procedure is presented for computing characteristics of Tollmien–Schlichting (T-S) waves in the course of their downstream evolution. It is based on the Global Stability Analysis of steady solutions of the full compressible Navier-Stokes equations and, therefore, does not have the restrictions associated with the parallel or quasi-parallel flow assumptions used in the classical methods of the linear stability analysis based on the boundary layer approximation. Hence, the methodology may be applied not only to simple boundary layers on smooth surfaces but also to non-parallel flows, e.g. those over surfaces with irregularities (steps, gaps, etc.). The developed procedure is validated by the comparison of the computed distribution of the T-S amplification factor (N-factor) in the zero pressure gradient boundary layer with the similar distribution computed based on the solution of the Orr-Sommerfeld equation and is shown to be accurate and robust.

Keywords: Global stability analysis, Tollmien-Schlichting waves, Boundary layer

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Материалы конференции
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Применение глобального анализа устойчивости для расчета характеристик волн Толлмина-Шлихтинга в пограничном слое на плоской пластине

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Аннотация. Представлена методика численного расчета характеристик волн Толлмина-Шлихтинга, распространяющихся вдоль пограничного слоя. Она основана на глобальном анализе устойчивости стационарных решений полных сжимаемых уравнений Навье-Стокса и потому не имеет ограничений, связанных с параллельностью или квазитрехмерностью потока, используемых в классических методах линейного анализа устойчивости уравнений пограничного слоя. Представленная методика может быть применена не только к простым пограничным слоям на гладких поверхностях, но и к непараллельным течениям, например, на поверхностях с уступами, кавернами и т.д. Точность и численная устойчивость разработанной процедуры верифицирована при помощи сравнения распределений N-факторов для волн Толлмина-Шлихтинга

в пограничном слое без градиента давления с распределениями, посчитанными по решениям уравнения Орра-Зоммерфельда.

Ключевые слова: глобальный анализ устойчивости, волны Толлмина-Шлихтинга, пограничный слой

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Introduction

Development of T-S waves is a key mechanism of convective instability of wall-bounded laminar flows, which plays a crucial role in the natural laminar-turbulence transition. This has stimulated numerous experimental, theoretical, and numerical studies of the T-S waves (see, e.g. monographs [1, 2] and a review paper [3]). In this paper we present first results of the project funded by the Russian Scientific Foundation and devoted to the development of a general methodology for predicting of the T-S waves evolution based on the Global Stability Analysis (GSA) of steady solutions of the full compressible Navier-Stokes (N-S) equations. Unlike the existing methods, the proposed methodology is applicable not only to simple boundary layers forming on smooth surfaces (parallel and quasi-parallel flows), but also to essentially non-parallel flows. It presents a three-stage numerical procedure. In the first stage, numerical solution of the steady N-S equations is obtained for the flow which stability is analyzed, i.e., the “baseflow” is defined. In the second stage, GSA of the baseflow is conducted which outcome is a set of complex eigenvalues and corresponding eigenvectors. Imaginary parts of the eigenvalues present the frequencies and the real part – the growth or decay (depending on the sign) rates of the small disturbances, while the real parts of the corresponding eigenvectors define the spatial shape of the disturbances. Finally, the third stage of the procedure consists in post-processing of the results of GSA, which allows defining streamwise distribution of the T-S waves amplification factor (N-factor), characterizing the growth rate of their amplitude in the course of downstream propagations.

The paper is organized as follows. Section 2 presents a brief overview of the methodology. Section 3 contains an example of its application to the predicting the T-S waves evolution in the zero pressure gradient boundary layer (ZPG BL), namely, the corresponding problem statement, some numerical details, results of the computations and their comparison with those of the classic 1D linear stability analysis. Finally, Section 4 summarizes major results of the study and presents its outlook.

Overview of the methodology

For the numerical integration of the compressible N-S equations performed in the first stage of the proposed procedure, an in-house CFD solver is used. It employs an implicit finite-volume formulation on structured multi-block overlapping grids. For approximation of the inviscid fluxes in the compressible N-S equations, the third-order upwind-biased scheme of Roe [4] is used, while the viscous fluxes are approximated with the second-order central scheme. The solver uses local time-stepping, which provides an iterative procedure for obtaining a steady solution, if it exists. In order to damp unsteadiness, the time integration is carried out with the use of a large time step (large Courant–Friedrichs–Lewy number), which is enabled by the use of an implicit scheme.

In the second stage, the GSA is conducted of the deeply converged (the maximum non-

dimensional residual is less than 10^{-7}) steady N-S solution obtained in the first stage. This is done with the use of the software earlier developed for GSA in the works [5, 6]. It is based on solution of the linear equations for the small perturbations, which are derived from the unsteady N-S equations. Unlike the original Roe scheme, the numerical scheme used for these linear equations employs the simple upwind finite-difference approximations based on the sign of the cell-face normal component of the baseflow velocity.

Finally, a post-processing of the GSA results is performed, which includes the following steps.

First, the complex eigenvectors are filtered with the use of the Kolmogorov-Zurbenko filter [7]. Then, streamwise distributions of the T-S waves amplitude, $A(x)$, is computed. It is defined as a local (at a given x) maximum amplitude of the filtered eigenvectors: $A(x) = \max_y\{A(x,y)\}$. Then, extracting is carried out of the streamwise distribution of the wavelength of the T-S waves, $\lambda(x)$, which is defined as the difference of the streamwise coordinates of the neighbouring maximums, $x_{\max}^{k+1} - x_{\max}^{k}$, in the real part of the eigenvector. After that, the group velocity of the T-S waves is calculated with the use of the relation $U_g = d\omega_i / d\alpha$, where ω_i is the frequency of the disturbances from the GSA and $\alpha = 2\pi/\lambda$. Finally, the last step of the post-processing consists in computing the streamwise variation of the amplitude of the running T-S wave, $B(x)$, and of its amplification factor $N(x)$. These parameters are computed based on the following consideration.

Let $\Delta x = x\text{-}x_0$ be the distance run by T-S waves for the time interval Δt. Then, the spatial decay-rate of these waves amplitude during this interval constitutes $D = \exp(\omega_i \Delta t)$. Assuming that the group velocity does not depend on x, the normalized distribution of $B(x)/B(x_0)$ accounting for this decay may be computed as: $B(x)/B(x_0) = A(x)/A(x_0)\cdot\exp(\omega_i \Delta x/U_g)$. This, in turn, allows a direct computation of the N-factor defined as $N = \ln[B(x)/B_{\min}]$, where $B_{\min}$ is the minimum value of $B(x)$ within the considered interval.

Application to ZPG BL

In order to assess robustness and accuracy of the methodology briefly outlined above it has been applied to the flow over a flat plate (zero pressure gradient boundary layer – ZPG BL).

Problem statement and results of the baseflow computations

We consider a ZPG BL at the Mach number $M_\infty = 0.05$ and the Reynolds number $\mathrm{Re}_0 = L_0 U_\infty/\nu = 3\cdot10^6$, where L_0 is the distance from the plate leading edge to the end of the T-S instability region at non-dimensional frequency normalized by the viscous time $F = 10^6\omega_i/U_\infty^2\nu$ equal to 30 according to the 1D linear stability theory. Corresponding frequency normalized by the convective time used in the GSA $\overline{\omega}_i = \omega_i(U_\infty/L_0) \equiv F\cdot\mathrm{Re}_0/10^6 = 3F = 90$ (hereafter, the bar over ω_i is dropped).

The computations were carried out for the plate of the length $2L_0$. In order to mitigate the effect of the inflow boundary conditions on the baseflow solution which stability is analyzed by the GSA, the computational domain is extended by adding the inviscid region with the length of $0.2L_0$ upstream of the plate leading edge (see Fig. 1). This results in the total length of the domain equal to $2.2L_0$. The size of the domain in the plate-normal direction is set equal to $0.2L_0$, which corresponds to about 40 boundary layer thicknesses at the outflow of the domain.

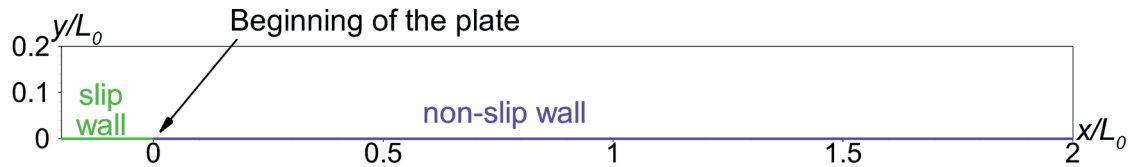

Fig. 1. Computational domain used for N-S computation of ZPG BL

The boundary conditions for the baseflow computations are imposed as follows.

At $y = 0$ and $x < 0$ the free slip (symmetry) conditions are specified, whereas at $y = 0$ and $x > 0$, the no-slip and non-permeability conditions for the velocity and the adiabatic conditions for the temperature are used.

At the free boundaries (inflow, outflow and upper ones) the characteristic boundary conditions are employed with the Riemann invariants defined by the free-flow parameters.

A size of the computational (x, y)-grid used in the computation is 10500×196 (2.058×10^6 cells total). The grid is gradually refined in -direction near the plate surface ($y = 0$) and in the x-direction near the leading edge of the plate ($x = 0$). Outside the regions with the refined grid, its streamwise step $\Delta x/L_0 = 2\cdot10^{-4}$ (e. g., at $F = 30$, this corresponds to about 100 points per wavelength) and the wall-normal one $\Delta y/L_0 = 5\cdot10^{-4}$. Note that this grid is actually designed for the GSA and is definitely excessive for the baseflow computation. However, this allows getting 100% grid-independent solution, on the one hand, and, on the other hand, permits avoiding interpolation of the baseflow solution on a finer GSA grid, which would be needed otherwise, with an insignificant penalty in terms of the additional CPU time because of the relatively low cost of the baseflow computation.

Figure 2 shows the baseflow velocity profile at $x/L_0 = 0.5$. One can see that it virtually coincides with the self-similar Blasius profile for the incompressible ZPG BL, which is not surprising for the considered low Mach number flow.

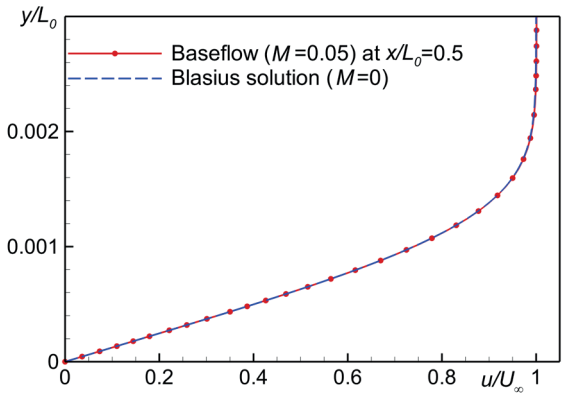

Fig. 2. Comparison of computed baseflow streamwise velocity profile at $x/L_0 = 0.5$ with self-similar Blasius solution

GSA problem setup and results

The problem setup has been defined based on results of a series of preliminary GSA computations with different sizes of the computational domain and two types of the boundary conditions (BCs) for the disturbances at its free boundaries, namely, Robin's conditions [8] and zero Dirichlet ones. These computations were aimed at finding a combination of the domain size and the BCs ensuring a minimum damage of results of the GSA caused by the approximate BCs. Their results (not shown) suggest that in this sense an optimal combination is the Dirichlet conditions imposed at the boundaries of the domain shown by red lines in Fig. 3. Its inlet and outlet are located at $x_1/L_0 = 0.15$ and $x_2/L_0 = 1.8$ respectively. According to the 1D linear stability theory, at the considered Reynolds number Re_0, this domain covers the entire range of the T-S instability for the frequencies within the range $40 \geq F \geq 20$ or $120 \geq \omega_i \geq 60$. The upper boundary of the domain is located at $y/L_0 = 0.12$.

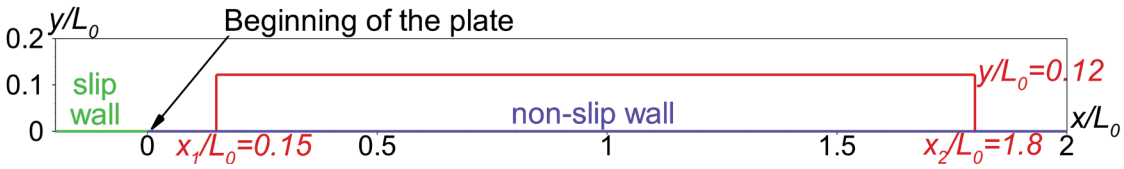

Fig. 3. Computational domain used in GSA of ZPG BL

Major results of the second stage of the proposed methodology, i.e., "raw" results of the GSA of the baseflow presented above are shown in Fig. 4 in the form of the growth rate – frequency map and of an example of the real part of the v-component of the eigenvector at $\omega_i = 90$ ($F = 30$).

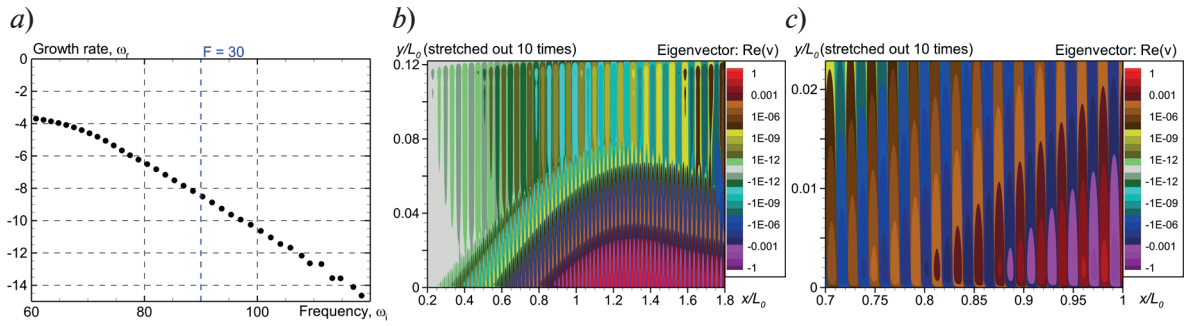

Fig. 4. Growth rate – frequency map (*a*), contours of real part of *v*-component of eigenvector corresponding to $\omega_i = 90$ (*b*), and its zoomed in fragment (*c*)

These results look quite reasonable and qualitatively similar to those available in the literature (see, e.g. [9, 10]).

We now move to the results of the last stage of the proposed procedure (GSA post-processing), which are presented in Fig. 5–7.

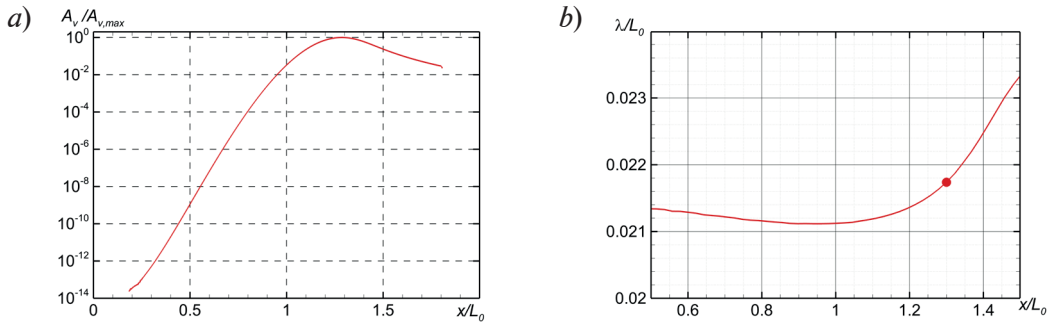

Fig. 5. Streamwise distributions of normalized amplitude of v-component of disturbances (*a*) and of T-S wavelength (*b*) at $\omega_i = 90$ ($F = 30$)

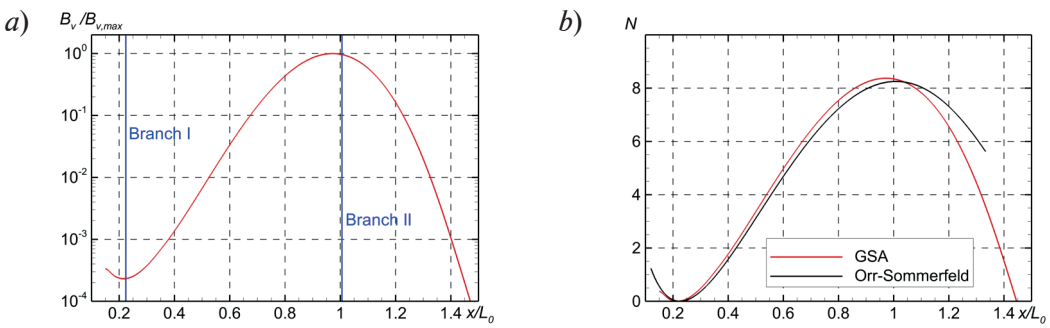

Fig. 6. Streamwise distribution of amplitude of running T-S wave with account of its decay (*a*) and its *N*-factor (*b*) at $F = 30$ ($\omega_i = 90$). Blue vertical lines show the instability boundaries on the first and second branches of T-S neutral curve according to Orr-Sommerfeld theory

In particular, the left frame of Fig. 5 shows the streamwise distributions of the normalized T-S waves amplitude $A_v(x) = A_{v,\max}$, where $A_{v,\max}$ is the maximum value of $A_v(x)$ reached at $x/L = 1.3$. The right frame of the figure depicts the plot of the streamwise distribution of the T-S wavelength $\lambda(x)$. One can see that the variation of the latter is marginal (about 7%). Considering this, for the further post-processing we use the value of $\lambda(x)$ at $x/L = 1.3$ where $A_v(x)$ reaches its maximum. This value is shown by the circle in the figure.

Given the wavelength dependence on the frequency is known, the group velocity of the T-S waves may be calculated as $U_g = d\omega_i/d\alpha$ (see Section 2). At $F = 30$, this gives $U_g = 0.36U_\infty$ which is close to the value of $U_g = 0.38U_\infty$ predicted by the 1D stability theory. This, in turn, allows computing the streamwise distribution of the amplitude of the running T-S wave with account of

its decay (see Section 2). An example of such distribution at $F = 30$ ($\omega_i = 90$) is shown in the left frame of Fig. 6, while the corresponding distribution of the T-S N-factor $N(x) = \ln[B_v(x)/B_{v,\min}]$ is presented in its right frame.

The figure also compares results of the present study with the similar Orr-Sommerfeld results for the incompressible flow. The comparison suggests very close agreement of the both approaches. This observation is supported by Fig. 7, which depicts a plot of N as the function of frequency. Thus, the results obtained are in good agreement with the theory, which indicates the reliability of the GSA itself and the developed technique as a whole.

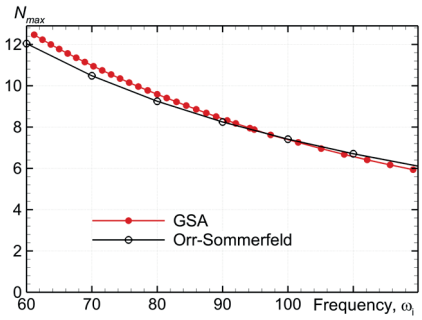

Fig. 7. Comparison of N-factor of T-S waves as function of frequency computed in the present study with Orr-Sommerfeld solution

Conclusion and outlook

The paper presents an outline of a general numerical methodology for predicting characteristics of the Tollmien–Schlichting waves based on the Global Stability Analysis of steady solutions of the full compressible Navier-Stokes equations. Unlike existing methods, this methodology does not rely upon the assumptions of parallel or quasi-parallel flow character, which opens a way to analyses of essentially non-parallel wall-bounded flows. It presents a three-stage procedure including 1) numerical solution of the steady Navier-Stokes equations for the flow in question, 2) Global Stability Analysis of this solution, and 3) post-processing of the results of this analysis aimed at extracting major characteristics of the Tollmien–Schlichting waves. Robustness and high accuracy of the proposed approach are demonstrated by its application to the canonic flat plate boundary layer, as an example: obtained results are in a good agreement with similar results of the classic linear stability analysis based on the Orr-Sommerfeld equation. This justifies applying the approach to studding the Tollmien–Schlichting waves in complex flows, particularly in the boundary layers on curved smooth surfaces and those with geometric irregularities (gaps, steps, etc.), which will be performed in the further work.

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REFERENCES

1. **Zhigulev V.N., Tumin A.M.,** Emergency of Turbulence, Nauka, Novosibirsk, 1987.
2. **Boiko A.V., Grek G.R., Dovgal A.V., Kozlov V.V.,** Physical Mechanisms of Transition to Turbulence in Open Flows, URSS Publishing Group, 2006.
3. **Theofilis V.,** Global linear instability. Annual Review Fluid Mech. (43) (2011) 319–352.
4. **Roe P.L.,** Approximate Riemann solvers, parameter vectors, and difference schemes. Journal of Comput. Phys. 2 (43) (1981) 357–372.
5. **Crouch J.D., Garbaruk A.V., Magidov D.,** Predicting the onset of flow unsteadiness based on global instability. Journal of Computational Physics. 2 (224) (2007) 924–940.
6. **Crouch J.D., Garbaruk A., Strelets. M.,** Global instability in the onset of transonic-wing buffet. Journal of Fluid Mechanics. (881) (2019) 3–22.
7. **Yang W., Zurbenko I.,** Kolmogorov–Zurbenko filters. WIREs Comp Stat. 2 (2010) 340–351.

8. **Gustafson K.,** Domain Decomposition, Operator Trigonometry, Robin Condition. Contemporary Mathematics. (218) (1998) 432–437.

9. **Ehrenstein U., Gallaire F.,** On two-dimensional temporal modes in spatially evolving open flows: the flat-plate boundary layer. Journal of Fluid Mechanics. (536) (2005) 209–218.

10. **Henningson D.S., Akervik E.,** The use of global modes to understand transition and perform flow control. Physics of Fluids. (20) (2008) 031302.

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