Original article
DOI: https://doi.org/10.18721/JPM.15211

TWO-FACTOR OPTIMIZATION IN THE BRACHISTOCHRONE PROBLEM

A. S. Smirnov [1,2]✉, *S. V. Suvorov* [3]

[1] Peter the Great St. Petersburg Polytechnic University, St. Petersburg, Russia;

[2] Institute of Problems of Mechanical Engineering of the Russian Academy of Sciences, St. Petersburg, Russia;

[3] Central Design Bureau of Transport Engineering, Tver, Russia

✉ smirnov.alexey.1994@gmail.com

Abstract. The paper puts forward a new modification of the well-known brachistochrone problem. The joint account of minimizing the motion time and the trajectory length in their functional relationship has been introduced. A two-factor optimization criterion (TOC) was constructed in the form of a product of two particular criteria, which made it possible to find the best compromise between them. On the TOC basis a solution to the problem of a two-factor brachistochrone was obtained using a preliminary consideration of the auxiliary problem on a brachistochrone with a given length. A rational practical solution of the problem was proposed. It was characterized by a simpler geometry than the strictly optimal one: to adopt a circular arc with a central angle selected on the basis of the taken TOC.

Keywords: brachistochrone, optimization, motion time, trajectory length, two-factor criterion, rational solution

For citation: Smirnov A. S., Suvorov S. V., Two-factor optimization in the brachistochrone problem, St. Petersburg Polytechnical State University Journal. Physics and Mathematics. 15 (2) (2022) 124–139. DOI: https://doi.org/10.18721/JPM.15211

Научная статья
УДК 531.332
DOI: https://doi.org/10.18721/JPM.15211

ДВУХФАКТОРНАЯ ОПТИМИЗАЦИЯ В ЗАДАЧЕ О БРАХИСТОХРОНЕ

А. С. Смирнов [1,2]✉, *С. В. Суворов* [3]

[1] Санкт-Петербургский политехнический университет Петра Великого, г. Санкт-Петербург, Россия;

[2] Институт проблем машиноведения Российской академии наук, г. Санкт-Петербург, Россия;

[3] Центральное конструкторское бюро транспортного машиностроения, г. Тверь, Россия

✉ smirnov.alexey.1994@gmail.com

Аннотация. В статье предлагается новая модификация известной задачи о брахистохроне. Введен совместный учет минимизаций времени движения и длины траектории в их функциональной взаимосвязи. Построен двухфакторный критерий оптимизации (ДКО) в виде произведения двух частных критериев, который позволил найти наилучший компромисс между ними; на основе ДКО получено решение задачи о двухфакторной брахистохроне с предварительным рассмотрением вспомогательной задачи о брахистохроне заданной длины. Предложено рациональное практическое решение задачи, обладающее более простой геометрией, чем строго оптимальное: принять дугу окружности с центральным углом, который подбирается на основе взятого ДКО.

Ключевые слова: брахистохрона, оптимизация, время движения, длина траектории, двухфакторный критерий, рациональное решение

Для цитирования: Смирнов А. С., Суворов С. В. Двухфакторная оптимизация в задаче о брахистохроне // Научно-технические ведомости СПбГПУ. Физико-математические науки. 2022. Т. 2 № .15. С. 124–139. DOI: https://doi.org/10.18721/ JPM.15211

Introduction

The problem of the brachistochrone was first formulated by Johann Bernoulli in 1696 (published in *Acta Eruditorum* under the title *Problema novum ad cujus solutionem Mathematici invitantur* [A new problem to whose solution mathematicians are invited]).

The problem was posed as follows:

Given in a vertical plane two points A and B, assign to the moving [body] M, the path AMB, by means of which — descending by its own weight and beginning to be moved [by gravity] from point A — it would arrive at the other point B in the shortest time.

This problem was successfully solved by such great scientists as Gottfried Leibniz, Johann Bernoulli, Guillaume de l'Hфpital and Isaac Newton [1]. Even though the solutions presented were different, the final answers turned out to be the same: the trajectory that is sought is a cycloidal arc. Importantly, the solution proposed by Bernoulli was the first step towards a new field in mathematical analysis, coming to be known as calculus of variations. Exploration of the brachistochrone problem has extended to modern times. In particular, the obtained solution has long been known for its practical applications in construction in tropical countries, where fast runoff of water from the roof considerably affects the building's durability throughout the rainfall season. For instance, the roofs of Buddhist pagodas are distinctly similar in shape to the cycloidal arc.

The classical problem of the brachistochrone is interesting both from an educational and a research perspective; it has been formulated as multiple generalizations, useful for wider practical applications. The first (but far from only) generalization was the problem of the brachistochrone in a resisting medium considered by Leonhard Euler. This direction has been continued in a series of modern studies [2—5], adopting models of both viscous and Coulomb friction. It is also intriguing to explore the problem of the brachistochrone for a rolling disk [6—8] and its different spatial configurations [9, 10], as well as some other generalizations (a detailed list is given in [11]).

Problem statement

Suppose that the start and end points A and B lie on the same horizontal level and are located at a distance l from each other (Fig. 1).

The problem of the brachistochrone can be interpreted as that on optimal design of an underground tunnel, whose classical statement only minimizes a single factor that is the motion time T of a point particle along the curve $y(x)$. Meanwhile, the length L of this curve turns out to be sufficiently large, which can be inconvenient for generating such a trajectory in practice. Moreover, it is often impossible to construct a tunnel precisely along the cycloid if there are underground rivers, so this configuration should be discarded in favor of other options. Finally, minimization of the path length is directly related to such economic metrics as the costs for building and subsequently maintaining a tunnel whose path is simulated by the required trajectory.

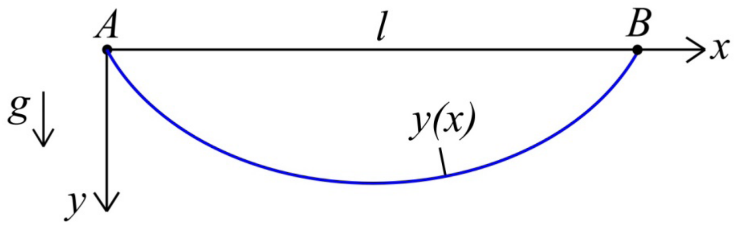

Fig. 1. Statement of the classical brachistochrone problem:
body M, acted on by its own gravity, should pass the trajectory $y(x)$ from point A to point B
in the shortest possible time (g is the acceleration of gravity)

It follows from the above that it is crucial to minimize the trajectory length L from a practical standpoint. However, simultaneously minimizing the two quantities T and L is meaningless, since they are determined by the known expressions

$$T = \int_0^l \sqrt{\frac{1+y'^2}{2gy}}dx, \quad L = \int_0^l \sqrt{1+y'^2}\,dx, \tag{1}$$

(the prime indicates a derivative with respect to the coordinate x), clearly taking their minimum values at different extremals $y(x)$ (cycloid and straight line, respectively).

Nevertheless, the problem on finding the best compromise between these two factors, achievable by constructing and analyzing an adequate two-factor optimization criterion, turns out to be fairly meaningful. Evidently, to obtain a trajectory with maximum efficiency, it seems reasonable to strive for the best relative compromise between the quantities T and L, taking into account their functional relationship. It is easy to understand that for this purpose it is advisable to synthesize a multiplicative optimization criterion in the form of the following composition of the particular criteria T and L:

$$J = T \cdot L = \min. \tag{2}$$

A similar criterion was successfully applied in other multicriterial mechanical problems [12, 13], where it proved to be effective, gaining major practical significance. Criterion (2) allows to estimate the degree to which the motion time T should be increased for the trajectory length L to decrease to the greatest extent compared to the increase in time. This is precisely what is meant by the best relative compromise between these factors.

The primary goal of this study is to analyze the two-factor optimization criterion (2) and find the optimal trajectory with respect to this criterion.

Determining the brachistochrone of the given length

Before we can focus on criterion (2), let us discuss in detail the auxiliary problem on finding a brachistochrone of a given length, which is mathematically formulated as follows [14]:

$$T = \min, \ L = \text{fix}, \tag{3}$$

where expressions (1) should be taken into account.

Problem (3) is an isoperimetric problem of variational calculus that should be solved by composing a function

$$H = \sqrt{\frac{1+y'^2}{2gy}} + \lambda\sqrt{1+y'^2} = H(y, y'), \tag{4}$$

where λ is a constant.

Next, we should consider the problem on the extremals of a functional with the integrand $H(y,y')$. As in the classical brachistochrone problem, this function does not explicitly depend on x, so we solve it using the first integral of the Euler–Lagrange equation:

$$H - y'\frac{\partial H}{\partial y'} = \left(\frac{1}{\sqrt{2gy}} + \lambda\right)\frac{1}{\sqrt{1+y'^2}} = C, \tag{5}$$

where C is a constant.

Let us introduce a standard substitution $y' = \text{ctg}\ \varphi$ in this equation, so that after some simplifications we obtain:

$$y = \frac{a\sin^2\varphi}{2(1-b\sin\varphi)^2}, \ a = \frac{1}{gC^2}, \ b = \frac{\lambda}{C}, \tag{6}$$

where a, b are the new constants related to C and λ.

Next, calculating y' by expression (6) and taking into account that $y' = \text{ctg}\ \varphi$, we obtain the following equation:

$$y' = \frac{a\sin\varphi\cos\varphi}{(1-b\sin\varphi)^3}\varphi' = \text{ctg}\ \varphi. \tag{7}$$

Separating the variables in it, we obtain:

$$\frac{a\sin^2\varphi}{(1-b\sin\varphi)^3}d\varphi = dx,\ x = a\int\frac{\sin^2\varphi}{(1-b\sin\varphi)^3}d\varphi. \tag{8}$$

To calculate the resulting integral, we use the following trigonometric substitution:

$$z = \mathrm{tg}\frac{\varphi}{2},\ \varphi = 2\mathrm{arctg}\, z,\ d\varphi = \frac{2dz}{1+z^2},\ \sin\varphi = \frac{2z}{1+z^2}. \tag{9}$$

As a result, after the transformations, we proceed to calculate the integral of the rational function:

$$\int\frac{\sin^2\varphi}{(1-b\sin\varphi)^3}d\varphi = 8\int\frac{z^2}{\left(z^2-2bz+1\right)^3}dz. \tag{10}$$

Let us use tables of integrals from [15] for this purpose. It is clear from the tables that the integral has different representations at $|b| < 1$ and $|b| > 1$. This is because the roots of the quadratic trinomial $z^2 - 2bz + 1$ are complex conjugate in the first case, and real in the second case.

Representations of integral (10). Let us consider the first case when $|b| < 1$. The integral of rational function (10) is in this case

$$\int\frac{z^2}{\left(z^2-2bz+1\right)^3}dz = \frac{1}{4(1-b^2)}\left[\frac{\left(2b^2-1\right)z-b}{\left(z^2-2bz+1\right)^2}+\right.$$
$$\left.+\frac{2b^2+1}{2(1-b^2)}\left(\frac{z-b}{z^2-2bz+1}+\frac{1}{\sqrt{1-b^2}}\mathrm{arctg}\frac{z-b}{\sqrt{1-b^2}}\right)\right]. \tag{11}$$

Returning to the initial variable φ, we obtain from equality (10) for $x(\varphi)$:

$$x = \frac{a}{(1-b^2)^2}\left[\frac{2b^2+1}{\sqrt{1-b^2}}\mathrm{arctg}\frac{\mathrm{tg}\frac{\varphi}{2}-b}{\sqrt{1-b^2}}+\frac{\cos\varphi\left(\left(4b^2-1\right)\sin\varphi-3b\right)}{2\left(1-b\sin\varphi\right)^2}\right]+x_0, \tag{12}$$

where x_0 is the integration constant.

As the problem is formulated so that the points A and B lie on the same horizontal at a distance l away from each other, then $y(0) = 0$, $y(l) = 0$. As evident from expression (6), these points correspond to the parameter values $\varphi = 0$ and $\pi = 0$. The constant x_0 is defined from the condition $x = 0$ at $\varphi = \pi$:

$$x_0 = \frac{a}{(1-b^2)^2}\left(\frac{2b^2+1}{\sqrt{1-b^2}}\mathrm{arctg}\frac{b}{\sqrt{1-b^2}}+\frac{3b}{2}\right). \tag{13}$$

In turn, the constant a is found from the condition $x = l$ at $\varphi = \pi$:

$$a = \frac{l(1-b^2)^2}{\frac{2b^2+1}{\sqrt{1-b^2}}\left(\frac{\pi}{2}+\mathrm{arctg}\frac{b}{\sqrt{1-b^2}}\right)+3b}. \tag{14}$$

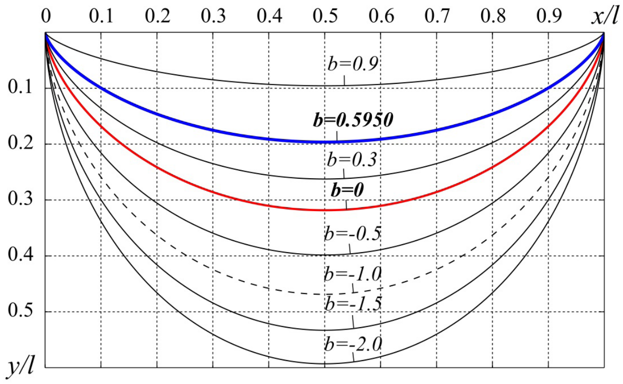

Fig. 2. Family of brachistochrones with different lengths (constructed in dimensionless coordinates)
See Eq. (15), case $0 < |b| < 1$; Eq. (16), case $b = 0$ (curve highlighted in red);
Eq. (26), case $b < -1$, and Eq. (32), case $b = -1$ (dashed curve).
The optimal trajectory with respect to the two-factor criterion (2) is highlighted in blue

As a result, the solution of problem (3) for the case $|b| < 1$ takes the form:

$$\begin{cases} x = \dfrac{a}{(1-b^2)^2}\left[\dfrac{2b^2+1}{\sqrt{1-b^2}}\left(\operatorname{arctg}\dfrac{\operatorname{tg}\dfrac{\varphi}{2}-b}{\sqrt{1-b^2}}+\operatorname{arctg}\dfrac{b}{\sqrt{1-b^2}}\right)+\right. \\ \left.+\dfrac{\cos\varphi\left(\left(4b^2-1\right)\sin\varphi-3b\right)}{2\left(1-b\sin\varphi\right)^2}+\dfrac{3b}{2}\right] \\ y = \dfrac{a\sin^2\varphi}{2\left(1-b\sin\varphi\right)^2} \end{cases}, \tag{15}$$

where the quantity a is defined by Eq. (14).

Thus, parametric solution (15) can be used to construct a family of brachistochrones corresponding to different values of the parameter b provided that $|b| < 1$. These trajectories are shown in Fig. 2, where the dimensionless coordinates x/l and y/l are plotted along the axes for convenience.

We should note that the value $b = 0$ corresponds to a cycloidal trajectory. Indeed, the equation for the cycloid follows directly from expressions (15):

$$\begin{cases} x = \dfrac{a}{4}\left(2\varphi-\sin 2\varphi\right) = r(\psi-\sin\psi) \\ y = \dfrac{a}{4}\left(1-\cos 2\varphi\right) = r(1-\cos\psi) \end{cases}, \tag{16}$$

where $r = a/4$ is the radius of the rolling circle; $\psi = 2\varphi$ is its rotation angle, varying from 0 to 2π.

The values $0 < b < 1$ correspond to the trajectories lying above the cycloid, and the values $-1 < b < 0$ correspond to the trajectories below it. Evidently, the dimensionless parameter b uniquely corresponds to the length L of the curve, which is defined by the second formula in (1).

To establish this correspondence, we substitute the expression $y' = \operatorname{ctg}\varphi$ into the given formula, consequently obtaining:

$$L = a\int_0^{\pi} \frac{\sin\varphi}{(1 - b\sin\varphi)^3} d\varphi = 4a\int_0^{\infty} \frac{z(1+z^2)}{(z^2 - 2bz + 1)^3} dz, \tag{17}$$

where the same substitution as above is performed for the variable (9).

Again, using the tables of integrals from [15], we find:

$$L = \frac{a}{(1-b^2)^2}\left[\frac{3b}{\sqrt{1-b^2}}\left(\frac{\pi}{2} + \mathrm{arctg}\frac{b}{\sqrt{1-b^2}}\right) + 2 + b^2\right]. \tag{18}$$

Substituting the value of a here, in accordance with Eq. (14) and introducing the dimensionless quantity $\delta = l/L$, which lies within $0 < \delta < 1$ based on the physical meaning of the problem, we obtain:

$$\delta = \frac{\left(\frac{\pi}{2} + \mathrm{arctg}\frac{b}{\sqrt{1-b^2}}\right)\frac{2b^2+1}{\sqrt{1-b^2}} + 3b}{\left(\frac{\pi}{2} + \mathrm{arctg}\frac{b}{\sqrt{1-b^2}}\right)\frac{3b}{\sqrt{1-b^2}} + 2 + b^2}. \tag{19}$$

This formula can be used to plot the dependence $\delta(b)$ over the interval $-1 < b < 1$ (Fig. 3). Notably, the value $b = 1$ corresponds to a straight line when $\delta = 1$, and the value $b = 0$ corresponds to a cycloid when $\delta = \pi/4 \approx 0.7854$

Finally, let us express the time of motion along the optimal trajectory according to the first formula in (1), performing the substitution $y' = \mathrm{ctg}\,\varphi$ in it and using expressions (6) and (8):

$$T = \sqrt{\frac{a}{g}}\int_0^{\pi} \frac{d\varphi}{(1 - b\sin\varphi)^2} = 2\sqrt{\frac{a}{g}}\int_0^{\infty} \frac{1+z^2}{(z^2 - 2bz + 1)^2} dz. \tag{20}$$

Again, using the tables of integrals from [15], we find:

$$T = 2\sqrt{\frac{a}{g}}\frac{1}{1-b^2}\left[\frac{1}{\sqrt{1-b^2}}\left(\frac{\pi}{2} + \mathrm{arctg}\frac{b}{\sqrt{1-b^2}}\right) + b\right], \tag{21}$$

where it should be borne in mind that the quantity a is determined by Eq. (14).

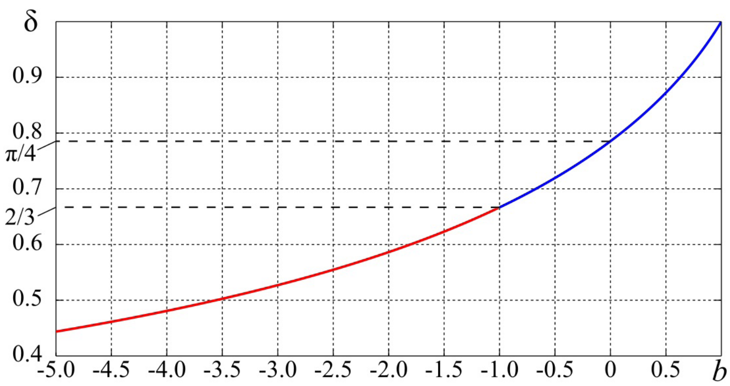

Fig. 3. Quantity δ as a function of the parameter b over the interval $-1 < b < 1$ (curve highlighted in blue) and at $b \le -1$ (curve highlighted in red)

Let us now consider the second case, when $|b| > 1$. Using the tables of integrals from [15], we confirm that integral (10) takes the form:

$$\int \frac{z^2}{\left(z^2-2bz+1\right)^3}dz = \frac{1}{4(1-b^2)}\left[\frac{\left(2b^2-1\right)z-b}{\left(z^2-2bz+1\right)^2}+\right.$$
$$\left.+\frac{2b^2+1}{2(1-b^2)}\left(\frac{z-b}{z^2-2bz+1}+\frac{1}{2\sqrt{b^2-1}}\ln\left|\frac{z-b-\sqrt{b^2-1}}{z-b+\sqrt{b^2-1}}\right|\right)\right]. \tag{22}$$

Returning to the initial variable φ, we obtain the function $x(\varphi)$ in accordance with Eq. (10):

$$x = \frac{a}{(1-b^2)^2}\left[\frac{2b^2+1}{2\sqrt{b^2-1}}\ln\left|\frac{\operatorname{tg}\frac{\varphi}{2}-b-\sqrt{b^2-1}}{\operatorname{tg}\frac{\varphi}{2}-b+\sqrt{b^2-1}}\right|+\frac{\cos\varphi\left(\left(4b^2-1\right)\sin\varphi-3b\right)}{2\left(1-b\sin\varphi\right)^2}\right]+x_0. \tag{23}$$

As before, we define the constant x_0 from the condition $x = 0$ at $\varphi = 0$:

$$x_0 = \frac{a}{(1-b^2)^2}\left(\frac{3b}{2}-\frac{2b^2+1}{2\sqrt{b^2-1}}\ln\left|\frac{b+\sqrt{b^2-1}}{b-\sqrt{b^2-1}}\right|\right). \tag{24}$$

Then we find the constant a from the condition $x = l$ at $\varphi = \pi$:

$$a = \frac{l(1-b^2)^2}{\frac{2b^2+1}{2\sqrt{b^2-1}}\ln\left|\frac{b-\sqrt{b^2-1}}{b+\sqrt{b^2-1}}\right|+3b}. \tag{25}$$

As a result, the solution of the problem for the case $|b| > 1$ takes final form:

$$\begin{cases} x = \dfrac{a}{(1-b^2)^2}\left[\dfrac{2b^2+1}{2\sqrt{b^2-1}}\ln\left|\dfrac{\left(\operatorname{tg}\frac{\varphi}{2}-b-\sqrt{b^2-1}\right)\left(b-\sqrt{b^2+1}\right)}{\left(\operatorname{tg}\frac{\varphi}{2}-b+\sqrt{b^2-1}\right)\left(b+\sqrt{b^2+1}\right)}\right|+\right. \\ \left.+\dfrac{\cos\varphi\left(\left(4b^2-1\right)\sin\varphi-3b\right)}{2\left(1-b\sin\varphi\right)^2}+\dfrac{3b}{2}\right] \\ y = \dfrac{a\sin^2\varphi}{2\left(1-b\sin\varphi\right)^2} \end{cases}, \tag{26}$$

where a is found from Eq. (25).

Parametric solution (26) can be used to construct the optimal trajectories corresponding to the values of the parameter b from the interval $b < -1$, which (see Fig. 2) continue the family of trajectories previously constructed for the values $-1 < b < 1$. As for the values $b > 1$, they are physically impossible, i.e., they do not correspond to the optimal trajectories. This can be better observed by expressing the curve length L by the second formula in (1):

$$L = \frac{a}{(1-b^2)^2}\left(-\frac{3b}{2\sqrt{b^2-1}}\ln\left|\frac{b+\sqrt{b^2-1}}{b-\sqrt{b^2-1}}\right| + 2 + b^2\right). \tag{27}$$

Substituting expression (25) here, we express the value of the dimensionless quantity $\delta = l/L$:

$$\delta = \frac{\dfrac{2b^2+1}{2\sqrt{b^2-1}}\ln\left|\dfrac{b-\sqrt{b^2-1}}{b+\sqrt{b^2-1}}\right| + 3b}{-\dfrac{3b}{2\sqrt{b^2-1}}\ln\left|\dfrac{b+\sqrt{b^2-1}}{b-\sqrt{b^2-1}}\right| + 2 + b^2}, \tag{28}$$

which, provided that $b > 1$, corresponds to negative values of the quantity δ, which cannot be the case in reality if we adopt the representation of δ as a ratio of lengths. Dependence (28) for the values $b < -1$ is also shown in Fig. 3.

The remaining step is to express the time of motion along the optimal trajectory in accordance with the first formula in (1) for the case under consideration:

$$T = 2\sqrt{\frac{a}{g}}\frac{1}{1-b^2}\left[-\frac{1}{2\sqrt{1-b^2}}\ln\left|\frac{b+\sqrt{b^2-1}}{b-\sqrt{b^2-1}}\right| + b\right], \tag{29}$$

where it should be borne in mind that the quantity a is determined by Eq. (25).

Notably, if $b \to 1$, we obtain, in accordance with Eq. (28), that $\delta \to 1$, (i.e., $L \to l$), while in accordance with Eq. (29), we obtain that $T \to \infty$, so that the case $b = 1$ is a limiting one.

To complete the picture, let us separately focus on the case when $b = -1$. This case is intermediate between the cases $-1 < b < 1$ and $b < -1$ analyzed above. Here, we obtain the following:

$$\int\frac{z^2}{\left(z^2-2bz+1\right)^3}dz = \int\frac{z^2}{\left(z+1\right)^6}dz = -\frac{10z^2+5z+1}{30(z+1)^5}. \tag{30}$$

Returning to the initial variable φ, we obtain in accordance with Eq. (10):

$$x = -\frac{4a\left(10\mathrm{tg}^2\dfrac{\varphi}{2} + 5\mathrm{tg}\dfrac{\varphi}{2} + 1\right)}{15\left(\mathrm{tg}\dfrac{\varphi}{2}+1\right)^5} + x_0. \tag{31}$$

Given that $x = 0$ at $\varphi = 0$, and $x = l$ at $\varphi = \pi$, we find, in accordance with expression (31), that $x_0 = 4a/15$, and $a = 15l/4$, so in this case we obtain $x_0 = l$.

Thus, the solution for the case $b = -1$ can be written as

$$\begin{cases} x = \dfrac{4a}{15}\left[1 - \dfrac{10\mathrm{tg}^2\dfrac{\varphi}{2} + 5\mathrm{tg}\dfrac{\varphi}{2} + 1}{\left(\mathrm{tg}\dfrac{\varphi}{2} + 1\right)^5}\right]. \\ y = \dfrac{a\sin^2\varphi}{2(1 - b\sin\varphi)^2} \end{cases} \tag{32}$$

This trajectory is shown in Fig. 2 by a dashed line. Finally, let us express the quantities L and T corresponding to this trajectory:

$$L = 4a\int_0^\infty \frac{z(1+z^2)}{(z+1)^6}dz = \frac{2a}{5},\ T = 2\sqrt{\frac{a}{g}}\int_0^\infty \frac{1+z^2}{(z+1)^4}dz = \frac{4}{3}\sqrt{\frac{a}{g}}. \tag{33}$$

It follows then that if $b = -1$, we obtain the value $\delta = l/L = 2/3 \approx 0.6667$, which is fully consistent with the graph shown in Fig. 3.

Thus, the problem on finding a brachistochrone with a given length can be considered solved.

Determining the optimal solution with respect to the two-factor criterion

Let us now proceed to search for the optimal solution by the multiplicative criterion (2). Apparently, in this case it is sufficient to use the solution to the above problem about the brachistochrone with the given length, where, provided that the value of L was known, a curve with the minimum possible motion T along it was found. This is because all other curves, yielding a larger result with respect to time for the given L, are also clearly worse with respect to the criterion (2), so they can be ignored performing the two-factor optimization procedure.

Thus, since we previously completed the first stage of optimization, we now consider criterion (2) only for the curves with the minimum motion time at the given length. Therefore, the problem is no longer variational but rather a conventional algebraic one on finding the minimum point of a single-variable function.

Considering the limiting cases, it is easy to see that criterion (2) indeed allows finding a specific optimal trajectory. As a matter of fact, we obtain for the first limiting case when the trajectory profile is close to rectilinear ($b \to 1$): $T \to \infty$, $L \to l$, i.e., $J \to \infty$. Conversely, if the trajectory has a deep profile ($b \to -\infty$), then $T \to \infty$, $L \to \infty$, so we once more obtain $J \to \infty$. This means that the criterion J, convenient to be considered as a function of the parameter b, should have an internal extremum, specifically, a minimum, in the interval $b < 1$. Recall that the time T is given by Eqs. (21) and (29) for cases $-1 < b < 1$ and $b < -1$, respectively, the length L is determined by expressions (18) and (27), and the parameter a included in these expressions is found from relations (14) and (25).

We introduce the dimensionless quantity I, proportional to the criterion J and related to it by the following formula:

$$I = \frac{J}{l}\sqrt{\frac{g}{l}}. \tag{34}$$

Because the representations for T and L are rather cumbersome, it is the easiest to determine the minimum point of function (34) by plotting its dependence on the parameter b (Fig. 4).

We can determine from the graph in Fig. 4 that the required minimum corresponds to the value $b_* = 0.5950$. We should note that its location in the interval $0 < b < 1$ can be also identified from general considerations. Indeed, the value $b = 0$ corresponds to the cycloidal profile, and the relative compromise for criterion (2) can only be reached by moving upwards from the cycloid, increasing the motion time T and decreasing the trajectory length L. It then becomes apparent what the values of T and L are equal to for motion along a trajectory that is optimal with respect to the two-factor criterion, as well as the value of the criterion J itself. If we substitute the value of b_* found into Eqs. (21) and (18), then, by virtue of (14) bearing in mind that $I_* = 2.9430$ (see Fig. 4), and using Eq. (34), we obtain:

$$T_* = 2.6265\sqrt{\frac{l}{g}},\ L_* = 1.1205l,\ J_* = I_* l\sqrt{\frac{l}{g}} = 2.9430l\sqrt{\frac{l}{g}}. \tag{35}$$

At the same time, we obtain for the cycloidal trajectory at $b = 0$:

$$T_c = \sqrt{2\pi}\sqrt{\frac{l}{g}} \approx 2.5066\sqrt{\frac{l}{g}},\ L_c = \frac{4l}{\pi} \approx 1.2732l. \tag{36}$$

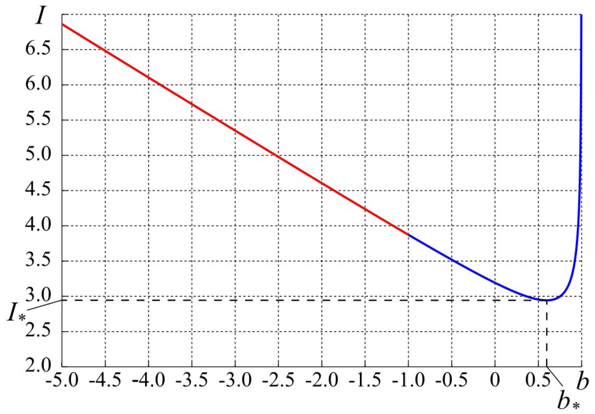

Fig. 4. Two-factor criterion I as a function of the parameter b in the interval $-1 < b < 1$ (curve highlighted in blue)and at $b \leq -1$ (curve highlighted in red). The minimum on the curve $I_* = 2.9430$ at $b_* = 0.5950$ is shown

Comparing the corresponding values of (35) and (36), we can conclude that the motion time for the optimal trajectory with respect to criterion (2) is 4.8% longer than that for the cycloid, and the length of this trajectory is smaller than that of the cycloid by 12%.

The obtained results clearly demonstrate the required best relative compromise between the two criteria T and L and prove that a fairly small increase in the motion time can yield a much greater decrease in the trajectory length. Based on the values obtained, we can once again recommend the criteria of form (2) for solving multicriterial problems in different areas of mechanics. Notice that the optimal trajectory found is also shown in Fig. 2 ($b = 0.5950$).

Constructing a rational solution

The solutions to most optimization problems in mechanics are optimal only in a formal (i.e., purely mathematical) sense, since their geometry is rather complex. Naturally, their practical implementations remain challenging. A related issue is to construct a solution that is not strictly optimal but has a simpler geometry and is more convenient for specific practical purposes. This solution can be called quasi-optimal, or rational; it is not optimal in general but rather in a class of functions characterizing the simplified geometry of the problem [16].

It is preferable to adopt a circular profile in the brachistochrone problem considered, that is, to consider trajectories in the form of a circular arc. Interestingly, Galileo proved that the shortest path is not always the fastest by comparing the motion time over a straight line with the motion time over a circular arc [11]. Thus, the problem on finding a rational solution is posed as follows:

Considering all the circles passing through the two given points A and B lying on the same horizontal (in our presentation), we are going to choose the one such that motion along it affords an extremum value to the given optimization criterion.

The formulated problem statement is illustrated graphically in Fig. 5.

We consider both the minimization for the motion time only and the two-factor criterion (2). If comparing the parameters of these circular trajectories with the characteristics of the solutions discussed above reveals sufficiently small differences, these rational solutions can be recommended for practical applications instead of the initial, strictly optimal solutions.

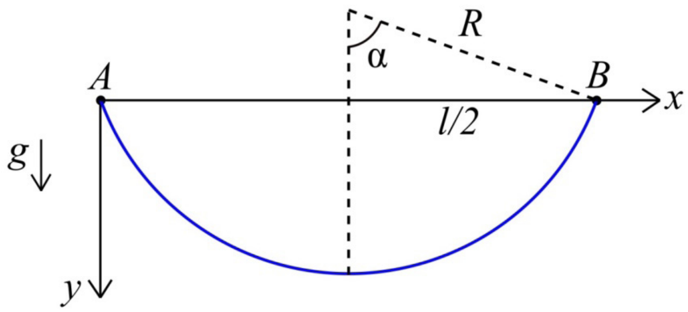

Fig. 5. Problem statement for the rational solution based
on the circular profile with the radius R (2α is the central angle)

It is well known that that the oscillation half-period of a mathematical pendulum, equal to the time of motion along the circumference from point A to point B, is found by Eq. [17]:

$$T = 2\sqrt{\frac{R}{g}}K\left(\sin\frac{\alpha}{2}\right), \tag{37}$$

where R is the circle radius, α is the oscillation amplitude, $K(\kappa)$ is the complete elliptic first-kind integral with the modulus κ.

It is clear that there is the quantities R, α and l are related in the following manner (see Fig. 5):

$$R = \frac{l}{2\sin\alpha}. \tag{38}$$

Substituting expression (38) into Eq. (37), we obtain the final expression for the motion time along a circular arc with the central angle of 2α:

$$T = \sqrt{\frac{2l}{g}}\frac{1}{\sqrt{\sin\alpha}}K\left(\sin\frac{\alpha}{2}\right) = T(\alpha). \tag{39}$$

Let us first consider the problem on minimizing the motion time. For this purpose, we differentiate function (39) with respect to the variable α, taking into account the rules for calculating derivatives of elliptical integrals, and equate the resulting expression to zero. As a result, the following equation can be obtained after the transformations:

$$(1+2\cos\alpha)K\left(\sin\frac{\alpha}{2}\right) = 2E\left(\sin\frac{\alpha}{2}\right), \tag{40}$$

where $E(\kappa)$ is a complete elliptic second-kind integral with the modulus κ.

The only root of equation (40) that corresponds to the meaning of the problem is $\alpha_{*1} = 1.2433$, while the motion time corresponding to it, found from Eq. (39), and the trajectory length, found by the formula $L = 2\alpha R = \alpha l/\sin\alpha$ and necessary for further comparisons, are equal to, respectively:

$$T_{*1} = 2.5233\sqrt{\frac{l}{g}}, L_{*1} = 1.3131l. \tag{41}$$

Evidently, the time T_{*1} exceeds the motion time along the cycloid, found by the first formula in (36), by only 0.7%. This means that the found circular arc with the central angle $2\alpha_{*1} = 2.4866$ (or 142.47°) is largely equivalent to the cycloid with respect to the time factor, so it can be recommended for practical applications.

Now, proceeding to find the best parameter of the circular trajectory with respect to the two-factor criterion (2), let us compose for it an expression accounting for Eq. (39) and bearing in mind that $L = \alpha l/\sin\alpha$:

$$J = l\sqrt{\frac{2l}{g}}\frac{\alpha}{\sin^{3/2}\alpha}K\left(\sin\frac{\alpha}{2}\right) = J(\alpha). \tag{42}$$

Differentiating this function with respect to α and the resulting expression equating to zero, we obtain the following equation after simplifications:

$$\left[\alpha(1+4\cos\alpha)-2\sin\alpha\right]K\left(\sin\frac{\alpha}{2}\right) = 2\alpha E\left(\sin\frac{\alpha}{2}\right), \tag{43}$$

whose only root consistent with the meaning of the problem is $\alpha_{*2} = 0.8720$. The motion time, trajectory length and two-factor criterion value corresponding to this root then take the following form:

$$T_{*2} = 2.6650\sqrt{\frac{l}{g}},\ L_{*2} = 1.1390l,\ J_{*2} = 3.0354l\sqrt{\frac{l}{g}}. \tag{44}$$

Clearly, the motion time increased by 5.6% compared to Eqs. (41), while the trajectory length decreased by 13.3%. These values also illustrate the best compromise reached between these factors for the case when optimization is carried out for a class of circular arcs. The last stage is to compare the found expressions (44) with similar values for the strictly optimal solution previously obtained with respect to the two-factor criterion, which are given by Eqs. (35).

For example, the time T_{*2} is only 1.5% larger than T_*, the length L_{*2} is only 1.7% larger than L_*, and finally, the value of the two-factor criterion J_{*2} is only 3.1% larger than J_*, which can be considered excellent results. Therefore, if a two-factor optimization criterion has to be used, the optimal trajectory with respect to this criterion, found above, can be replaced with acceptable accuracy by a circular arc with the central angle $2\alpha_{*2} = 1.7440$ (or 99.92°), which has much simpler geometry. The circular profiles found, which are optimal with respect to the above criteria T = min and J = min, are shown in Fig. 6 by dashed lines together with the corresponding trajectories that are generally optimal with respect to the same criteria, represented by solid lines.

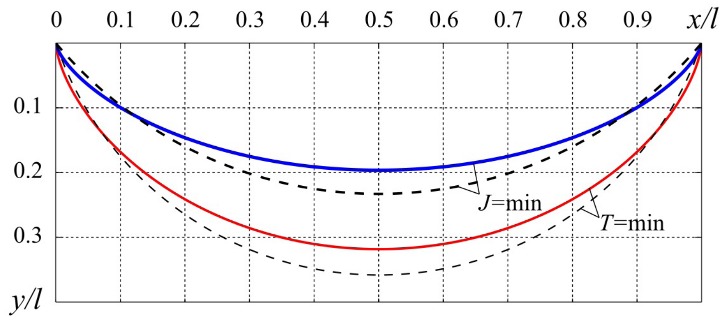

Fig. 6. Comparison of optimal profiles (solid lines) with rational ones (dashes); *T*, *J* are optimization criteria (the motion time of a point particle along the curve and the multiplicative criterion, respectively)

Conclusion

We have proposed a modification of the classical brachistochrone problem, allowing for minimizing the length of trajectory in addition to minimizing the time of motion. The problem was solved by constructing a two-factor multiplicative optimization criterion. As we analyzed the problem posed, we have considered in detail the problem on the brachistochrone of a given length; the results provided the simplest way to finding the optimal trajectory with respect to the two-factor criterion adopted. The numerical values presented in the paper and the comparisons drawn lead us to conclude that such multiplicative criteria are satisfactory, so they can be used for solving other problems on optimization of mechanical systems, where a relatively optimal balance between several factors should be reached.

Furthermore, we have constructed a rational solution which is characterized by simplified geometry and is easy to use.

We have established that the circular profile is virtually equivalent to the strictly optimal solution if the profile's parameters are selected properly based on the optimization criterion adopted. For this reason, we also recommend to construct similar rational solutions for many other problems.

REFERENCES

1. **Yakovlev V. I.** The beginnings of analytical mechanics, Institute of Computer Sciences, Moscow, Izhevsk, 2002 (in Russian).

2. **Golubev Yu. F.,** Brachistochrone with friction, Journal of Computer and Systems Sciences. 49 (5) (2010) 719–730.

3. **Zarodnyuk A. V., Cherkasov O. Yu.,** Brachistochrone with linear viscous friction, Moscow University Mechanics Bulletin. 70 (3) (2015) 70–74.

4. **Sumbatov A. S.,** Brachistochrone with Coulomb friction as the solution of an isoperimetrical variational problem, Intern. J. Non-Linear Mech. 88 (January) (2017) 135–141.

5. **Wensrich C. M.,** Evolutionary solutions to the brachistochrone problem with Coulomb friction, Mech. Res. Commun. 31 (2) (2004) 151–159.

6. **Legeza V. P.,** Brachistochrone for a rolling cylinder. Mechanics of Solids. 45 (1) (2010) 27–33.

7. **Akulenko L. D.,** An analog of the classical brachistochrone for a disk, Doklady Physics. 53 (3) (2008) 156–159.

8. **Sumbatov A. S.,** Problem of the brachistochronic motion of a heavy disk with dry friction, Intern. J. Non-Linear Mech. 99 (March) (2018) 295–301.

9. **Legeza V. P.,** Brachistochronic motion of a material point on a transcendental surface, Intern. Appl. Mech. 56 (3) (2020) 358–366.

10. **Gladkov S. O., Bogdanova S. B.,** The theory of a space brachistochrone, Tomsk State University Journal of Mathematics and Mechanics. 68 (2020) 53–60 (in Russian).

11. **Sumbatov A. S.,** The problem on a brachistochrone (classification of generalizations and some recent results), Trudy MFTI [Transactions of Moscow Institute of Physics and Technology]. 9 (3) (2017) 66–75 (in Russian).

12. **Smolnikov B. A., Smirnov A. S.,** Novyy kriteriy optimizatsii v zadache Gomana [A new optimization criterion in the Hohmann problem], In book: XII Vserossiyskiy Syezd po fundamentalnym problemam teoreticheskoy i prikladnoy mekhaniki. Sbornik trudov. V 4-kh tomakh. T. 1. Obshchaya i prikladnaya mekhanika [Transactions of "The 12-th All-Russian Congress on Fundamental Problems of Theoretical and Applied Mechanics", Aug. 19–24, 2019, Ufa, Republic of Bashkortostan, Russia. The collection of works in 4 Vols., Vol. 1: General and Applied Mechanics, Bash. State Univ. Reg. Inf. Center, Ufa (2019) 266–268 (in Russian).

13. **Smirnov A. S., Smolnikov B. A.,** Catenary optimization, In book: Trudy seminara «Kompyuternyye metody v mekhanike sploshnoy sredy» [Transactions of the seminar "Computer Methods in Continuum Mechanics"], 2019–2020, Saint-Petersburg University Publishing, St. Petersburg (2020) 35–50 (in Russian).

14. **Gladkov S. O., Bogdanova S. B.,** Analytical and numerical solution of the problem on brachistochrones in some general cases, J. Math. Sci. 245 (4) (2020) 528–537.

15. **Gradshteyn I. S., Ryzhik I. M.,** Table of integrals, series and products, Translated from Russian, Seventh Edition, Eds. Jeffrey A., Zwillinger D., Elsevier Inc., Amsterdam, Boston, Heidelberg, etc., 2007.

16. **Suvorov S. V., Smirnov A. S.,** Otsenka effektivnosti optimalnykh balochnykh konstruktsiy [Performance evaluation of the optimal beam structures], Nedelya nauki SPbPU: materialy nauchnoy konferentsii s mezhdunarodnym uchastiyem. Institut prikladnoy matematiki i mekhaniki [Proceedings of the scientific conference "Science Week at SPbPU" with international participation. The Institute of Applied Mathematics and Mechanics], November 18–23, 2019, St. Petersburg Polytechnic University Publishing, St. Petersburg (2019) 102–104 (in Russian).

17. **Sikorskiy Yu. S.,** Elementy teorii ellipticheskikh funktsiy s prilozheniyami k mekhanike [Elements of the theory of elliptic functions with applications to mechanics]. KomKniga, Moscow, 2006 (in Russian).

СПИСОК ЛИТЕРАТУРЫ

1. **Яковлев В. И.** Начала аналитической механики. М., Ижевск: Институт компьютерных исследований, 2002. 352 с.

2. **Голубев Ю. Ф.** Брахистохрона с трением // Известия Российской академии наук. Теория и системы управления. 2010. № 5. С. 41–52.

3. **Зароднюк А. В., Черкасов О. Ю.** К задаче о брахистохроне с линейным вязким трением // Вестник Московского университета. Сер. 1: Математика. Механика. 2015. № 3. С. 65–69.

4. **Sumbatov A. S.** Brachistochrone with Coulomb friction as the solution of an isoperimetrical variational problem // International Journal of Non-Linear Mechanics. 2017. Vol. 88. January. Pp. 135–141.

5. **Wensrich C. M.** Evolutionary solutions to the brachistochrone problem with Coulomb friction // Mechanics Research Communications. 2004. Vol. 31. No. 2. Pp. 151–159.

6. **Легеза В. П.** Брахистохрона для катящегося цилиндра // Известия Российской академии наук. Механика твердого тела. 2010. № 1. С. 34–41.

7. **Акуленко Л. Д.** Аналог классической брахистохроны для диска // Доклады Академии наук. 2008. Т. 419. № 2. С. 193–196.

8. **Sumbatov A. S.** Problem of the brachistochronic motion of a heavy disk with dry friction // International Journal of Non-Linear Mechanics. 2018. Vol. 99. March. Pp. 295–301.

9. **Legeza V. P.** Brachistochronic motion of a material point on a transcendental surface // International Applied Mechanics. 2020. Vol. 56. No. 3. Pp. 358–366.

10. **Гладков С. О., Богданова С. Б.** К теории пространственной брахистохроны // Вестник Томского государственного университета. Математика и механика. 2020. № 68. С. 53–60.

11. **Сумбатов А. С.** Задача о брахистохроне (классификация обобщений и некоторые новые результаты) // Труды МФТИ. 2017. Т. 9. № 3. С. 66–75.

12. **Смольников Б. А., Смирнов А. С.** Новый критерий оптимизации в задаче Гомана // XII Всероссийский съезд по фундаментальным проблемам теоретической и прикладной механики. 19–24 августа 2019 г., г. Уфа, Республика Башкортостан, Россия. Сборник трудов в 4-х тт. Т. 1. Общая и прикладная механика. Уфа: РИЦ БашГУ, 2019. С. 266–268.

13. **Смирнов А. С., Смольников Б. А.** Оптимизация цепной линии // Труды семинара «Компьютерные методы в механике сплошной среды» 2019–2020. СПб.: Изд-во Санкт-Петербургского университета, 2020. С. 35–50.

14. **Гладков С. О., Богданова С. Б.** Аналитическое и численное решение задачи о брахистохроне в некоторых общих случаях // Итоги науки и техники. Серия «Современная математика и ее приложения». Тематические обзоры. 2018. Т. 145. С. 114 – 122.

15. **Градштейн И. С., Рыжик И. М.** Таблицы интегралов, рядов и произведений. СПб.: БХВ-Петербург, 2011. 1232 с.

16. **Суворов С. В., Смирнов А. С.** Оценка эффективности оптимальных балочных конструкций // Неделя науки СПбПУ: материалы научной конференции с международным участием. Институт прикладной математики и механики. 18 – 23 ноября 2019 г., Санкт-Петербургский политехнический университет. СПб.: Изд-во Политехнического университета, 2019. С. 102–104.

17. **Сикорский Ю. С.** Элементы теории эллиптических функций с приложениями к механике. М.: КомКнига, 2006. 366 с.

THE AUTHORS

SMIRNOV Alexey S.
Peter the Great St. Petersburg Polytechnic University
Institute of Problems of Mechanical Engineering of the Russian Academy of Sciences
29 Politechnicheskaya St., St. Petersburg, 195251, Russia
smirnov.alexey.1994@gmail.com
ORCID: 0000-0002-6148-0322

SUVOROV Sergei V.
Central Design Bureau of Transport Engineering
45v, Peterburgskoe HWY, Tver, 170003, Russia
suvorovsv96@gmail.com
ORCID: 0000-0002-7461-2742

СВЕДЕНИЯ ОБ АВТОРАХ

СМИРНОВ Алексей Сергеевич – *ассистент Высшей школы механики и процессов управления Санкт-Петербургского политехнического университета Петра Великого, Санкт-Петербург, Россия; стажер-исследователь Института проблем машиноведения Российской академии наук.*
195251, Россия, г. Санкт-Петербург, Политехническая ул., 29
smirnov.alexey.1994@gmail.com
ORCID: 0000-0002-6148-0322

СУВОРОВ Сергей Викторович – *инженер по испытаниям 3-й категории Центрального конструкторского бюро транспортного машиностроения.*
170003, Россия, г. Тверь, Петербургское шоссе, 45в
suvorovsv96@gmail.com
ORCID: 0000-0002-7461-2742

Received 25.01.2022. Approved after reviewing 23.03.2022. Accepted 23.03.2022.
Статья поступила в редакцию 25.01.2022. Одобрена после рецензирования 23.03.2022. Принята 23.03.2022.