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THE DYNAMIC CHARACTERISTICS OF A RESONATOR OF THE GIROSCOPE BASED ON ELASTIC WAVES IN SOLIDS: FINITE-ELEMENT MODELING

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In the paper, the eigenfrequencies of a hemispherical resonator of the Coriolis vibratory gyroscope have been studied by the finite element method (FEM) using ANSYS Mechanical. Consideration was given to the feasibility of various FE used in the ANSYS to solve the problem of determining the eigenfrequencies. The specifics of working with shell and solid-state elements were established. The results of analytical and numerical solutions of the mentioned problem were compared. The presence of "mathematical" frequency split caused by the used FEM and the unsymmetrical mesh of the FEM was noted, and the need to take this split into account when introducing the defect distribution function into the model was pointed out. The technique for finding the frequency split value resulted by added defect in the presence of "mathematical" frequency split component was demonstrated.

Keywords: Coriolis vibratory gyroscopes, hemispherical resonator, eigenfrequency split, finite element method

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ПРИМЕНЕНИЕ КОНЕЧНО-ЭЛЕМЕНТНОГО МОДЕЛИРОВАНИЯ ДЛЯ ИССЛЕДОВАНИЯ ДИНАМИЧЕСКИХ ХАРАКТЕРИСТИК РЕЗОНАТОРА ТВГ

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В статье методом конечных элементов (МКЭ) исследованы собственные частоты полусферического резонатора твердотельного волнового гироскопа (ТВГ) с использованием программного комплекса ANSYS Mechanical. Рассмотрена применимость различных КЭ, использующихся в ANSYS, для решения задачи определения собственных частот. Установлены особенности работы с оболочечными и твердотельными элементами. Проведено сравнение результатов аналитического и численного решений задачи по определению собственных частот резонатора. Отмечено наличие «математического» расщепления частоты, вызванного применяемым МКЭ и несимметричностью КЭ-сетки, а также необходимость учета данного расщепления при внесении в модель функции распределения дефекта. Представлен способ нахождения величины расщепления от внесенного дефекта при наличии составляющей «математического» расщепления.

Ключевые слова: твердотельный волновой гироскоп, полусферический резонатор, расщепление частоты, метод конечных элементов

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Introduction

Various types of gyroscopes find their application in modern guidance, position control and stabilization systems. Gyroscopes are employed in shipbuilding, air- and spacecraft industries, as well as rocket engineering. Production of Coriolis vibratory gyroscopes (CVGs) is a promising direction of gyroscopic technology development. New generations of optic, vibratory, wave solid-state and other gyroscopes are taking over the traditional gyroscopic devices [1, 2]. Manufacture and application of the new types of gyroscopes are associated with the need for miniaturization alongside with meeting the accuracy, reliability and service life requirements.

Effect of inertia of elastic waves is the cornerstone of modern CVGs operation [3]. When the object with a CVG installed rotates, the device registers the standing wave precession emerging due to constant oscillations of the sensory organ – resonator. Measuring the angular displacement of the wave makes it possible to calculate the angular velocity in inertial space used in production of angular velocity and displacement sensors [4]. The CVG design often includes thin cylindrical and hemispherical resonators, which are classic shells with mode shapes convenient for application.

We further consider a hemispherical resonator design (Fig. 1), its primary geometrical parameters, as well as the properties of its material (Table 1).

Resonator Q-factor is one of the main properties characterizing the device operation. Therefore, low viscosity materials are used for the production of resonators. Fused quartz is a material with one of the lowest levels of viscosity. For instance, metals have the respective parameter at a 2–3 orders higher level, thus producing resonators with inferior performance specifications [4], and, consequently, fail to provide high accuracy. Note that fused quartz possesses the same isotropy of elastic characteristics, which is essential for the material of CVG sensory organs. We selected the values of physical and mechanical characteristics of fused quartz taken for the calculations presented in the article in compliance with the state standard GOST 15130-86.

There are strict accuracy requirements imposed upon gyroscopic systems, particularly on CVGs. Currently, due to rather high development of electronic devices, the factors defining the accuracy of a CVG are geometrical and physical properties of its elastic element (the resonator), which are obtained in the process of its production. In other words, the accuracy parameters of CVGs are most influenced by various errors resulting from the production of the elastic element (circumferentially varying thickness, out-of-roundness, surface roughness, circumferentially varying thickness of the metallic film, etc.), as well as by imperfection of the physical characteristics of the material applied

Table 1

Main characteristics of the hemispherical resonator

Material	Fused quartz KU-1
Elastic modulus, GPa	73.6
Poisson's ratio	0.17
Density, kg/m^3	2210
Outer radius of hemisphere, mm	15.25
Wall thickness of hemisphere, mm	0.90

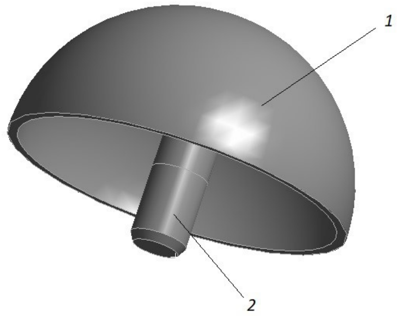

Fig. 1. Geometrical model of the hemispherical resonator: *1* – thin hemispherical shell, *2* – stem

(anisoelasticity, anisodamping, inhomogeneous density, internal defects, etc.). The indicated imperfections cause the effect of eigenfrequencies and modes split in the resonator due to its axial symmetry perturbation. The effect manifests itself in the non-ideal resonator spectrum obtaining two closely spaced frequencies instead of one with two close eigenmodes excitation, which leads to a change in the operation regime of the device. With its operation frequency splitting, the resonator Q-factor decreases, which leads to a drift of the gyroscope, and consequently, to a drop of the CVG accuracy characteristics. To estimate the influence of the errors and defects of the CVG manufacturing process on its operating frequency split value, the design stage includes various mathematical methods, one of them being the finite element method (FEM).

Many works are devoted to the approaches of calculating the eigenvalues of thin shells of different shape, as well as to the studies on the eigenfrequency split (see, for example, articles [5–8] and thesis [11]). Along with the widespread analytical calculations, the authors of the mentioned sources also use FEM. For instance, paper [8] reports good, compared to the analytical methods, agreement of the results obtained via FEM. Noteworthy is that the listed works, with the exception of [11], do not mention the resonator eigenfrequency split caused by the application of FEM itself and apparently attributable to the error in the method of calculating the eigenfrequencies (it is the block Lanczos method [9, 10] in the present paper) and imperfection of the FE mesh. We will further denote this split as "mathematical".

Out of all considered papers, thesis [11] provides the most detailed study of the resonator frequency split using FEM. The author notes that it is impossible to divide the "mathematical" (in [11] referred to as "parasitic") split from the one caused by manufacturing defects and proposes to minimize the value of the "mathematical" split by means of constructing a finite element mesh according to the author's methodology. However, just like the other indicated works, the thesis pays insufficient attention to the influence of the phase angle between the harmonics of various defects while investigating their simultaneous impact on the resonator frequency split. Noteworthy, the authors often describe a change in the wall thickness of the resonator as harmonic function with respect to the middle surface, while technologically, in the manufacturing process, the outside surface of the resonator is usually of better quality than that of the inside one. Therefore, describing the change of the thickness via a circumferential harmonic function with respect to coordinate surface of the resonator formed by the outside radius of the hemisphere is of certain interest as well.

The purpose of the present paper is to produce a finite-element model (FEM model) of the CVG resonator designed to determine the values of the operating eigenfrequency of the resonator with sufficient accuracy and verify a possibility of taking into account circumferentially varying thickness of the resonator wall in the frequency split calculations.

To build a FEM model, we used ANSYS Mechanical software [12]. In this paper, we considered two separate modeling problems:

calculation of the exact value of the operating eigenfrequency of the resonator;

estimation of the influence of various factors on the resonator operating eigenfrequency split, for example, circumferentially varying thickness.

Problem statement

The performed study originated from a problem of calculating the oscillation eigenfrequencies of a thin hemispherical shell. It is appropriate to apply Hamilton's variational principle [13]:

$$\delta I = \\ = \delta\int_{t_0}^{t_1} L\left(q_1, \ldots, q_n,\ \dot{q}_1, \ldots, \dot{q}_n,\ t\right) dt = 0,$$

where δI is the change in the desired functional, $L = T - W$ (T, W are kinetic energy of the studied elementary volume of the shell and potential energy of elastic strains respectively).

We can write the kinetic and potential energy expressions in the general form as

$$T = \frac{1}{2}\int_\sigma \rho V^2 d\sigma,$$

$$W = \frac{1}{2}\int_\sigma \left(\sigma_{11}\varepsilon_{11} + \sigma_{22}\varepsilon_{22} + \sigma_{33}\varepsilon_{33} + \right. \\ \left. + \sigma_{12}\varepsilon_{12} + \sigma_{13}\varepsilon_{13} + \sigma_{23}\varepsilon_{23}\right) d\sigma,$$

where ρ, kg/cm^3, is the material density; $\mathbf{V}$, m/s^2, is the absolute velocity vector of an arbitrary point of the elastic body; σ_{11}, σ_{22}, σ_{33}, Pa, are normal stresses of the specified element of the elastic body; ε_{11}, ε_{22}, ε_{33} are the respective normal strains; σ_{12}, σ_{13}, σ_{23}, Pa, are shear stresses of the specified element of the elastic body; ε_{12}, ε_{13}, ε_{23} are the respective shear strains; σ, m^3, is the volume of the specified element of the elastic body.

The expressions for T and W applicable to the calculations of the resonator eigenfrequency can be found in a number of works; however, the expressions can differ in the fact that either stress or strain tensor components are neglected in them. Thus, in this work, in order to compare the expected frequency values obtained via FEM, we used the expressions in the formulation of the thin shell theory presented, in particular, in book [14]:

$$T = \frac{1}{2}\int_0^{2\pi}\int_0^{\frac{\pi}{2}} \rho h V^2 A_1 A_2 d\theta d\varphi,$$

$$W = \frac{Eh}{2\left(1-\nu^2\right)}\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\left[\left(\varepsilon_1 + \varepsilon_2\right)^2 - \right. \\ \left. - 2\left(1-\nu\right)\left(\varepsilon_1\varepsilon_2 - \left(\frac{\omega}{2}\right)^2\right)\right] A_1 A_2 d\theta d\varphi + \rightarrow \\ \rightarrow + \frac{Eh^3}{24\left(1-\nu^2\right)}\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\left[\left(\kappa_1 + \kappa_2\right)^2 - \right. \\ \left. - 2\left(1-\nu\right)\left(\kappa_1\kappa_2 - \tau^2\right)\right] A_1 A_2 d\theta d\varphi,$$

where h, m, is the thickness of the hemispherical shell; A_1, A_2 are Lamé parameters; ν is Poisson's ratio of the material; E, MPa, is the elastic modulus of the material; ε_1, ε_2 are parameters characterizing tensile strain of the mid-surface; κ_1, κ_2 are parameters characterizing bending strain of the mid-surface; ω, τ are parameters characterizing shear and torsional strains respectively; θ, φ, deg, – zenith and azimuth angles respectively.

The use of the Ritz method [15] reduces the eigenfrequencies problem to the algebraic eigenvalue problem:

$$\left(A - \lambda^2 B\right)\mathbf{C} = 0,$$

where A, B are matrices connected with the kinetic and potential matrices, as well as with the coordinate functions; $\mathbf{C}$ is a column-vector of unknown coefficients; $\boldsymbol{\lambda}$ is a column-vector of the eigenfrequencies values.

It is important to note that the relations in the shells theory formulation specified above comply with the main assumptions of the Kirchhoff – Love theory of thin shells [16]:

plane section normal to the mid-surface remains normal to the mid-surface after deformation;

normal stress along the axis normal to the mid-surface is not considered due to its smallness;

the thickness of the shell does not change during a deformation.

In addition, the calculation implies an assumption of small strains and, respectively, of neglecting the geometrical non-linearity.

The problem statement will further include not only Solid type elements, but also the Shell type elements. Modeling Shell type elements in ANSYS software also entails the abovementioned Kirchhoff – Love assumptions, with the exception of the first one. In this case, a deformation-related change in the angle between the plane section and the mid-surface is admissible. This formulation corresponds to the Mindlin – Reissner shell variant [17], which is better known as Timoshenko shell in Russian literature [18].

Selection of optimal resonator FE models

The operation of the resonator is characterized by the numerical value of the operating eigenfrequency and the respective mode shape. The values of the eigenfrequency and the mode shape are defined by the following factors:

Dimensions of the resonator (radius and thickness of the hemisphere);

dimensions and mounting mode of the stem;

physical and mechanical properties of the selected material.

Traditionally, the CVG operation is based on coupled vibrations in two gyroscopically connected elliptical modes corresponding to its operating frequency [4, 19] (Fig. 2).

Nonideality of the geometrical parameters of the resonator entails a perturbation of the axial symmetry, which causes the splitting effect of its oscillation eigenfrequencies. The effect manifests itself in the non-ideal resonator spectrum obtaining two closely spaced frequencies instead of one with two close eigenmodes excitation, which leads to a change in the operation regime of the device and to an unacceptable reduction in accuracy [7]. Therefore, the developed FE model needs to possess an appropriate sensitivity level to register the frequencies split corresponding to its admissible value for the product under study. In other words, to estimate the resonator characteristics which influence the accuracy parameters of CVG, we need to pay attention not only to finding the exact value of the oscillation eigenfrequency of the selected design, but more so to the dependence on the variation of its studied deviations from the ideal system.

Because of this, we propose applying various models to solve two separate problems: to find the exact operating eigenfrequency of the resonator oscillations and to determine how circumferentially varying thickness influence the frequency split value.

The required accuracy of the estimated eigenfrequency value (for the first problem) and the value of the frequency split depending on the defect dimensions (for the second problem) can serve as the criteria defining the quality of the developed models. Let us note that the accuracy of determining the eigenfrequency value of up to 1 Hz is sufficient for the primary analysis. We took the admissible error of the split calculation (for the second problem) equal to $1 \cdot 10^{-4}$ Hz, which is one order higher than the admissible value of frequency split for fused quarz hemispherical resonators after balancing.

Since we can use various types elements to build FE models, we compared the models built with some types of Shell and Solid elements. The objective was to find an optimal balance between

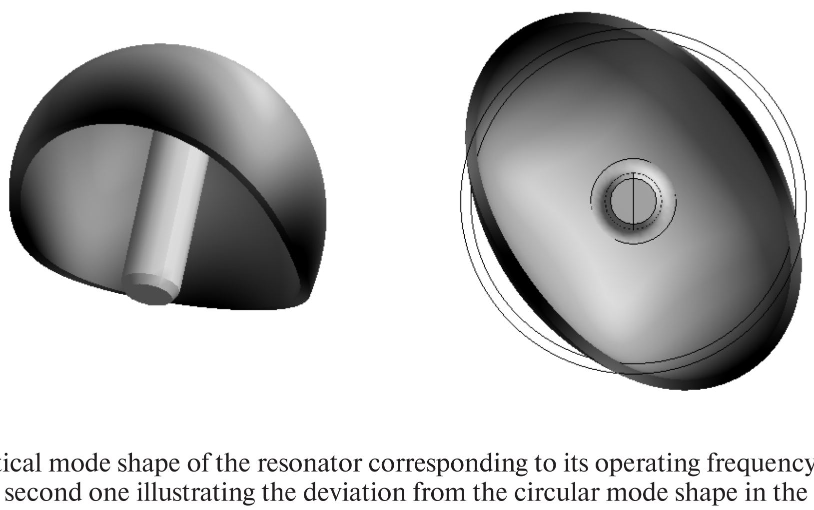

Fig. 2. Elliptical mode shape of the resonator corresponding to its operating frequency (two planes demonstrated, the second one illustrating the deviation from the circular mode shape in the oscillations plane)

the labor cost of the computation and its accuracy.

First model (FEM I). The hemisphere was meshed into SHELL181 type elements, i.e. into shell elements of the 1st order, which had 4 nodes with six degrees of freedom in each (linear translations in three axes and rotation about these axes);

Second model (FEM II). The hemisphere was meshed into SHELL281 type elements, i.e. into shell elements of the 2nd order, which had 8 nodes with six degrees of freedom in each (linear translations in three axes and rotation about these axes);

Third model (FEM III). The hemisphere was meshed into SOLID186 type elements, i.e. into solid elements of the 2nd order, which had 20 nodes with three degrees of freedom in each (linear translations in three axes);

Fourth model (FEM IV). The hemisphere was divided into SOLID187 type elements, i.e. into solid elements of the 2nd order, which had 10 nodes with three degrees of freedom in each (linear translations in three axes).

To improve the accuracy of the obtained results, we needed to meshed the initial geometry of the object into a regular finite-element mesh. Such a mesh is distinguished by its structured nature and the order of the predominantly regular-shaped elements used.

In all for FE models, we meshed the stem using SOLID185 elements of the 1st order to reduce the computational time, as the degree of meshing the resonator stem has no influence of the values of its eigenfrequencies corresponding to the second (elliptical) mode shape. We meshed the rounding area from the stem to the hemisphere using SOLID186 elements (Fig. 3). We should note that the absence of the indicated influence is due to the stem design, in particular, to its diameter and length. The chosen design parameters provide sufficient offset of the resonator frequencies caused by the bend of the stem from the operating elliptical frequency. In case of near values of the indicated frequencies, there may be negative effects considered in paper [20].

In the course of the calculations aimed at obtaining the model of the desired accuracy and minimal computational time, we set the minimal necessary number of the elements and nodes of the hemisphere. In case of shell models, the size

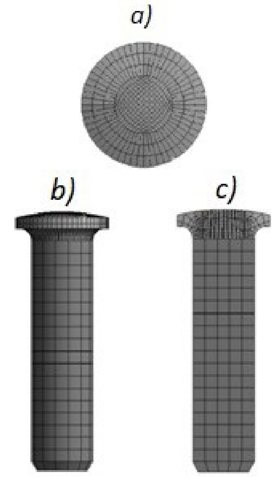

Fig. 3. Finite element mesh of rounding area (*a*) and resonator stem (*b*, *c*). Meshing into SOLID186 and SOLID185 elements respectively; *b*, *c* – the stem and its longitudinal section

of hemisphere elements varied in the range from 0.9000 to 0.1125 mm. The mesh of SHELL281 finite elements is shown in Fig. 4, *a*, *b*.

When studying the stress-strain state of the shell or plate type structural elements using SOLID type finite elements, to obtain acceptable result, we need to provide sufficient number of elements over the thickness in the FE model. Therefore, this paper considers different options of the model with the number of the elements over the thickness ranging from one to eight. Fig. 4, *c*, *d* shows an example of meshing the hemisphere into SOLID186 elements.

Fig. 5, 6 shows dependences of the resonator eigenfrequency values on the number of nodes employed in the models. The obtained diagrams help determine the optimal number of nodes which provide the value of Δ, the change on the value of the obtained solution with the growing number of nodes, less than 0.01%.

Analysis of the diagrams in Fig. 5 corresponding to the models with shell elements allows us to conclude that the value of the eigenfrequency sets at the level of 4808–4809 Hz. The diagrams are given for the model with the mid-surface as a reference plane. In FEM I, $\Delta = 0.005$ %, while in FEM II, II $\Delta = 0.006$ %. The difference of the obtained frequency values between two models reaches no more than 0.6 Hz. We can see here,

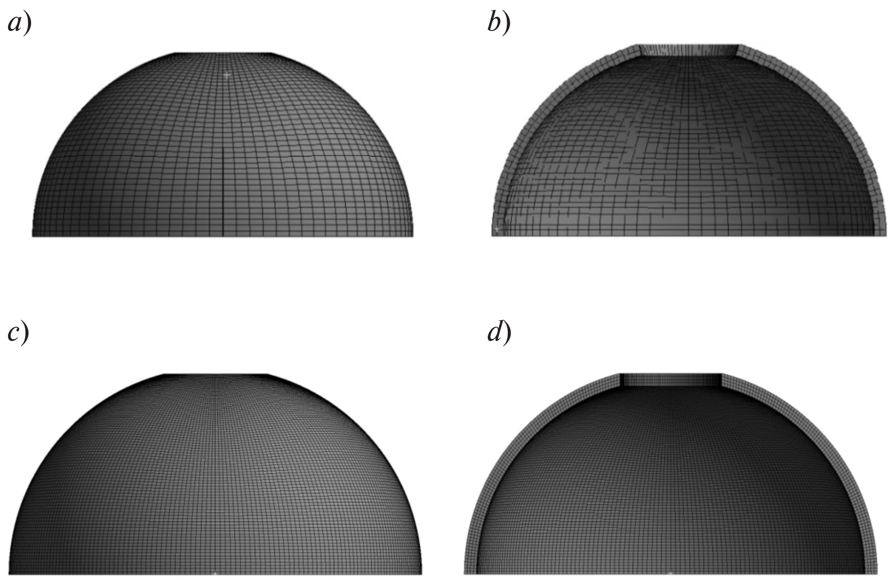

Fig. 4. Meshing of hemisphere (*a*, *c*) and its cross-section (*b*, *d*) into SHELL281 (*a*, *b*) and SOLID186 (*c*, *d*) elements

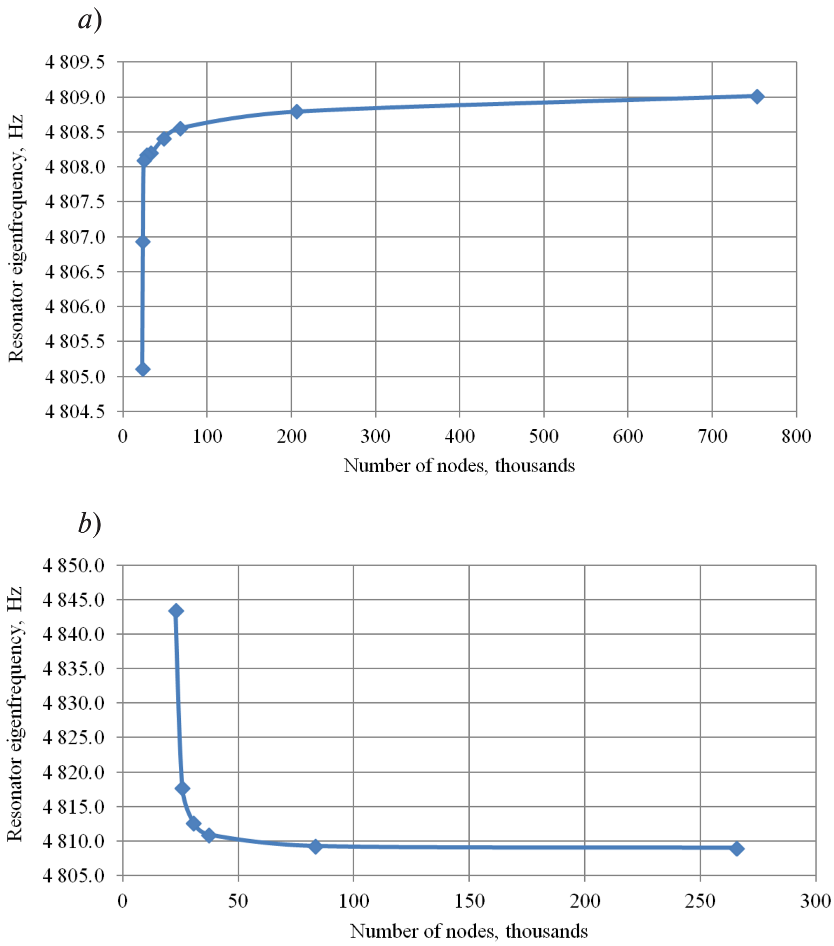

Fig. 5. Computational diagrams of resonator eigenfrequencies dependences on the number of nodes applied in FEM I (SHELL281) (*a*) and FEM II (SHELL181) (*b*)

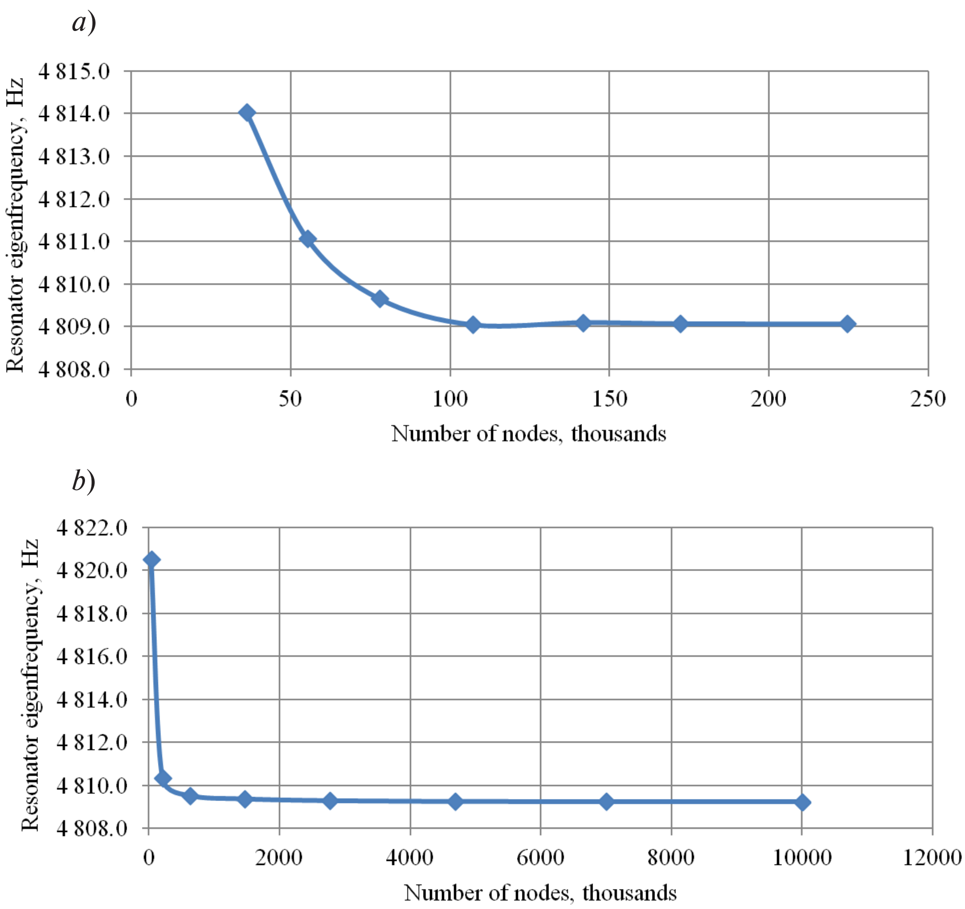

Fig. 6. Computational dependences similar to the ones in Fig. 5, but for FEM III (SOLID186) (*a*) and FEM IV (SOLID187) (*b*)

that the use of the 2nd order elements (SHELL281) is preferable, as we need only 50 thousand nodes for the steady-state solution in this case, while no less than 83 thousand nodes are required for the 1st order elements.

Note that convergence of the results to the steady-state value of eigenfrequency in the models using shell elements occurs from different directions (from the highest/lowest value to the steady-state value). The problem with the SHELL181 type elements converges to a steady-state solution from the greater values, which corresponds to the classical behavior of the numerical problem convergence diagrams. However, the use of SHELL281 elements produces a reverse effect. This is probably due to the applied type of the contact interaction (SHELL-SOLID) of the hemisphere and the stem. The bodies contact along a line, which leads to local loading along the faces of solid elements. Moreover, if we exclude the stem from the model and apply a boundary condition like a fixed support along the corresponding face of the hemisphere, we can observe the “stiff” characteristic of the diagram for SHELL281 element as well.

Analysis of the solid model diagrams (Fig. 6) shows that the desired eigenfrequency value remains constant starting from a certain level of the model discretization. The difference of the obtained frequency values between SOLID186 and SOLID187 elements is no more than 0.6 Hz. The values of Δ, obtained from mesh convergence, were 0.0002 % in both FEM III and FEM IV. However, to obtain the results close in accuracy, we required 1.5 times more FEM nodes in case of SOLID187 elements. Because of this, it is feasible to apply SOLID186 for computations. In general, we can conclude that meshing into four elements over the thickness of the hemisphere wall is optimal for the eigenfrequencies analysis providing the accuracy of up to 0.1 Hz. Note that for the primary estimate of the eigenfrequency values (up to 1 Hz accuracy), it is sufficient to divide the hemisphere thickness only into two elements. The difference from the steady-state solution in this case is 0.44 Hz.

Results of the operating resonator eigenfrequency calculations

Simultaneously with the finite elements methods, for a comparison we performed an analytical calculation using the simplified expressions for a hemisphere shell obtained by Rayleigh [21], and the Goldenweiser's thin shells theory [22] with specifications as to the tensile property of the mid-surface. The results of all the obtained values of the eigenfrequency in the second mode shape are given in Table 2.

Note that in the present paper, when comparing the obtained data, we took the result of the steady-state calculation using solid elements as the exact solution. This is due to the fact that the solid element models imply solving the elasticity theory problem without any simplifications. In addition, the accuracy of the result is defined by the degree of the model discretization and the mathematical error of the very method of solving the eigenvalues problem.

As we can see from Table 2, the result of determining the eigenfrequency of the resonator obtained using the considered shell and solid finite elements models shows comparable values. Moreover, among the calculation results obtained using analytical methods, the Goldenweiser's value is the most accurate one.

In the course of solving the second problem set above, i.e. to determine the split of the resonator eigenfrequencies, we used SHELL281 (FEM I) type elements to analyze the influence of circumferentially varying thickness on the dynamic behavior of the resonator. The use of shell elements allows us to set the change in the thickness of the hemisphere quite easily with no changes of its geometry, as well as to reduce the estimated time while saving the sufficient accuracy of the calculations.

Influence of defects on the value of the operating resonator frequency split

Features of introducing various defects into the resonator FEM. To determine the split of the operating frequency of the resonator in the presence of any defect, we need to introduce a distribution function of this defect into the model. In a real resonator, the distribution of such defects as circumferentially varying thickness and inhomogeneous density over the azimuth or zenith angle is random. However, researchers often use a harmonic dependence of the defect distribution in modeling as the simplest in respect of calculations:

$$x(\alpha) = x_0 + X \cdot \sin(m\alpha + \beta),$$

where x_0, m, is the nominal value of the parameter (hemisphere thickness, material density); X, m, is the amplitude; m is the number of the defect harmonic; α, β, deg, are the initial angle and phase respectively.

The number of the harmonic is chosen arbitrarily depending on the simulated defect function; nonetheless, let us note that the fourth

Table 2

Values of the operating eigenfrequency of the resonator oscillations (second mode) calculated via different methods

Calculation method	Frequency, Hz
Rayleigh	5277.60
Goldenweiser's thin shells	4814.20
Finite elements	
SHELL type	4809.02
SOLID type	4809.08

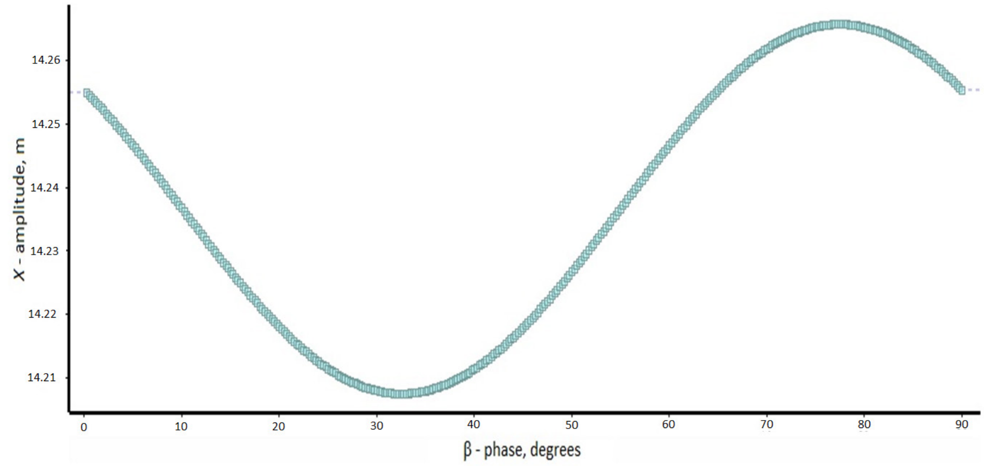

Fig. 7. Estimated dependence of the total operating frequency split of the resonator on phase angle of the defect function

harmonic has the most influence on the split of resonator frequency, which corresponds to the elliptical mode shape. Sources [3, 19] describe the reason of the indicated influence in detail. At the same time, the harmonics different from the fourth one have one order less impact. Thus, to model the worst case of a defect affecting the frequency split, it is expedient to use the fourth harmonic of the defect distribution.

As it was mentioned before, when we employ FEM, a "mathematical" split occurs even in ideal geometry. It is important to note, that this "mathematical" split is a measurable (calculating) variable. The main reason of its emergence is the non-ideality of the finite element mesh. Therefore, when we are solving the eigenfrequency problem in the resonator model and introduce a defect $x(\alpha)$, we obtain a certain total split, with the "mathematical" split being one of its components.

Based on the assumption that the operating eigenfrequency under study is elliptical, and consequently, corresponds to the second harmonic, the "mathematical" split is represented in the form of the second harmonic as a component of the total split. Given the above, to determine the value of the split components, we need to find the phase shift of the harmonics with respect to each other. Note that if the harmonics representing the "mathematical" split and the defect distribution coincide, the total split is a sum of the values of the split of both of the defects, while in case of the antiphase, it a difference thereof. The indicated feature is apparent, for example, if we introduce the fourth harmonic of the defect with a change in the initial phase angle from 0° to 90° (Fig. 7).

In this figure, we plotted the total split value $\Sigma\Delta f$ along the vertical axis and the phase angle β along the horizontal axis. The phase angle of the extremums depends on the built mesh and is preserved with the change in the amplitude and the nature of the defect (the defect is caused by the manufacturing errors or heterogeneity of the material).

Calculation of the split value when adding a defect into the resonator FEM. One of the factors defining the eigenfrequency split effect in the resonator is mass unbalance of the sensory organ. In a real hemispherical resonator, the unbalanced mass is continuously distributed across the whole shell, which in case of excitation causes oscillations of its center of mass and leads to the split of the frequencies and reduction in the Q-factor due to the oscillation energies dissipation in supports. The mass unbalance itself is due to the above mentioned geometrical and physical errors, including circumferentially varying thickness and inhomogeneous density. Since applying FE models leads to emergence of the "mathematical" split, to calculate the component of the introduced circumferentially varying thickness, we have to find the value of such split as well. It is possible, if we use a harmonic function as a defect (circumferentially varying thickness) distribution function:

$$h(\alpha) = h_0 + X_h \cdot \sin(m\alpha + \beta),$$

where h_0, m, is the nominal thickness of the resonator wall, X_h, m, is a half of the value of circumferentially varying thickness, α, deg, is an angular coordinate corresponding to the azimuth angle.

In this paper, to solve the problem of finding the resonator eigenvalues be means of ANSYS, the split of the operating frequency can be determined via the difference between two near frequencies corresponding to the elliptical mode shape. The calculation results in a certain total split of various defects, if they are introduced into the resonator FEM. In case of "ideal" geometry (absence of defects), the total split is equal to the "mathematical" split:

$$\sum \Delta f = \Delta f_m = f_{II}^{(2)} - f_{II}^{(1)},$$

where Δf_m, Hz, is the "mathematical" split; $f_{II}^{(2)}$, $f_{II}^{(1)}$, Hz, are the highest and lowest values of the eigenvalues defining the split.

If we introduce an additional defect into consideration, for example, circumferentially varying thickness, then the total split is determined as

$$\sum \Delta f = f_{II}^{(2)} - f_{II}^{(1)} = \\ = \Delta f_m \cdot \sin(2\alpha_1 + \beta_1) + \Delta f_h \cdot \sin(m\alpha_2 + \beta_2),$$

where Δf_h, Hz, is the split cause by circumferentially varying thickness; α_i, β_i, deg, are the angles and phases defining the defect distribution with respect to the azimuth angle.

The above relations show that unambiguous determination Δf_h required knowing the initial angles and phases of the harmonics corresponding to the defects. If we bear this in mind and use two extremal values of the total split value (when finding the harmonics in phase and antiphase), then we can compile a simple system of two equations allowing us to determine the splits from each defect:

$$\begin{cases} \sum \Delta f_{\max} = \Delta f_h + \Delta f_m, \\ \sum \Delta f_{\min} = \Delta f_h - \Delta f_m. \end{cases}$$

To automate the calculation of two extremal values of the total split in ANSYS, we can denote and as parameters, thus producing a diagram similar to the one presented in Fig. 7. Note that the found angle and phase corresponding to the "mathematical" split are maintained in the model even after a change in characteristics describing other defects, which excludes the need for additional calculation to provide the search for the extremal values of the total split. Therefore, using the block Lanczos method in ANSYS software allows us to determine the value of the split caused by circumferentially varying thickness by means of accounting for the "mathematical" split.

As an example of using the presented methodology of calculations, for the hemispherical resonator of the given design, we obtained numerical values of the splits caused by circumferentially varying thickness and inhomogeneous density (Table 1). Each effect was considered separately. The value of thickness variations amounted to 6 μm, the density of the fused quartz was ranging within 2200–2220 kg/cm^3. The defects function is represented in a form of the fourth harmonic.

As a result, the obtained values are limited by the applied method of numerical calculation and simplifications accepted in the mathematical model of the shell for the SHELL281 type finite elements.

In the process of the study, we noted a number of features. For example, when adding a density defect into the FEM, we established that the value of the "mathematical" split changes in proportion to the average arithmetical value of frequencies $f_{II}^{(1)}$ and $f_{II}^{(2)}$, which allows calculating the "mathematical" split value not only in the course of the computations, when the harmonics are in phase/antiphase. This does not eliminate the need for finding the extremal values of the total split for a newly developed FEM. At the same time, we do not observe any similar

Table 3

Results of calculating the operating resonator frequency split

Set defect function	Split, Hz
$\rho(\alpha) = 2210 + 10 \cdot \sin 4\alpha$	7.7193
$h(\alpha) = 9 \cdot 10^{-4} + 3 \cdot 10^{-6} \cdot \sin 4\alpha$	6.4881

dependence for a change, for example, in the thickness of the hemisphere or the elastic modulus of the material. Thus, seeking the dependence of the split on the defects changing the generated stiffness matrix of the FEM requires finding the new value of the “mathematical” split at each change as well.

Conclusion

The presented research produced finite element models (FEMs) including various types of elements, which can be used in the studies of resonator operation dynamics. We established that to determine the eigenfrequencies of a hemispherical resonator with elliptical oscillations, the use of SHELL281 elements is preferable, as it provides an optimal computational time/accuracy ratio. In addition, FEMs using the indicated elements and the methods described and tested in this paper are handy in the studies on the influence various defects exert on the resonator operation.

Sensitivity of shell elements to the circumferentially varying thickness is apparently limited by the error of the numerical calculation method; its estimation transcends the scope of this article. The described methodology allows studying a frequency split caused by the uneven distribution of the material properties (density, elastic modulus, Poisson’s ratio), as well as circumferentially varying thickness of the inside, outside, and middle surface of the hemisphere using standard functions of the ANSYS Mechanical software package.

The paper notes the importance of taking into account the phase of the functions describing the defects distribution due to the presence of the “mathematical” split in any FEM caused by the calculation method error and asymmetry of the finite element mesh.

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