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IDEAL FOCUSING SYSTEMS WITH HOMOGENOUS MAGNETIC FIELDS

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Cases of charged particle pack ideal focusing in the presence of constant magnetic field have been considered. One-directional ideal space-time focusing was shown to remain only on conditions that a magnetic field being homogenous and its direction being the same as the one of quadratic potential growth. Axially symmetric electrostatic fields with superimposed magnetic field were taken as an example because of their practical importance in the mass spectrometry. It was concluded that at least one equation with separated motion should be linear to maintain the ideal space-time focusing.

Keywords: spectrometry, ideal focusing, ion trap, homogenous magnetic field

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Introduction

Field structures providing ideal space-time focusing (IPTF) of a charged particle beam in one of the directions serve as a theoretical basis for synthesizing a wide class of high-resolution devices for mass-spectrometry [1, 8–11]. The entire range of electrostatic fields with a quadratic dependence of the potential on one of the coordinates was described for the first time in [2]. These fields are the ones providing IPVF for the beam. Electrostatic integrable systems with ideal space-time focusing in one of the directions have been sufficiently studied by now [3–7]; one of the non-integrable types considered closely are the so-called Cassinian traps [8–11], combining in some cases the functions of the trap and the ion bunch manipulation system [11].

There are limitless possibilities for synthesis of ideally focusing electrostatic systems. However, to complete the picture, the problem of preserving ISTF in the presence of a magnetic field should be reconsidered (see, for example, an earlier discussion in [12]).

Analysis of Ideal Focusing Systems

Let us first construct a dimensionless model of the problem posed. The motion of a charged particle in an electric field with the potential Φ and a magnetic field with a magnetic flux density vector $\mathbf{B}$ is described by the equation

$$m\frac{d^2\mathbf{R}}{dt^2} = -q\cdot\operatorname{grad}\Phi + q\frac{d\mathbf{R}}{dt}\times\mathbf{B}, \qquad (1)$$

where m, q are the mass and charge of the ion; $\mathbf{R}$ is the radius vector of the particle; t is the time.

We introduce the dimensionless variables $\mathbf{r}$, τ, φ, $\mathbf{b}$, μ in terms of the following relations:

$$\mathbf{R} = l\mathbf{r}, t = T\tau, \Phi = \Phi_0\phi,$$
$$\mathbf{B} = B_0\mathbf{b}, m = m_0\mu,$$

where l is the characteristic size of the system; Φ_0, B_0 are its characteristic potential and magnetic induction; T is the time unit; m_0 is the mass unit;

$$\mathbf{r} = (x, y, z), \mathbf{b} = (b_x, b_y, b_z)$$

are, respectively, the dimensionless radius vector and the dimensionless vector of the magnetic field, which is unit everywhere for a homogeneous field and unit at a given point in space for an inhomogeneous field.

Substituting new variables into Eq. (1), we obtain:

$$m_0\mu\cdot\frac{l}{T^2}\cdot\frac{d^2\mathbf{r}}{d\tau^2}=$$
$$=-q\cdot\frac{\Phi_0}{l}\cdot\frac{\partial\phi}{\partial\mathbf{r}}+q\cdot\frac{l}{T}\cdot\frac{d\mathbf{r}}{d\tau}\times(B_0\mathbf{b})$$

or

$$\mu\frac{d^2\mathbf{r}}{d\tau^2}=-q\cdot\frac{T^2\Phi_0}{m_0 l^2}\cdot\frac{\partial\phi}{\partial\mathbf{r}}+$$
$$+q\cdot\frac{TB_0}{m_0}\cdot\frac{d\mathbf{r}}{d\tau}\times\mathbf{b}.$$

One of the three cases has to be chosen taking T as a constant for measuring time: either the coefficient for the first or second term on the right-hand side is set to unity, or a certain relationship is imposed between the values of these coefficients.

Let us consider the first case, taking

$$T=l\sqrt{\frac{m_0}{|q\Phi_0|}}. \tag{2}$$

Then we obtain the equation of motion in the form

$$\mu\ddot{\mathbf{r}}=-\nabla\phi+\beta\cdot\dot{\mathbf{r}}\times\mathbf{b}, \tag{3}$$

where the overdot denotes differentiation with respect to τ, and the gradient is taken over the components of the vector $\mathbf{r}$. The parameter β is determined by the expression

$$\beta=lB_0\sqrt{\frac{1}{m_0}\left|\frac{q}{\Phi_0}\right|}. \tag{4}$$

Evidently, $\mathbf{b}=\mathbf{e}_z$ for a homogeneous magnetic field directed along the axis z. This scheme for introducing dimensionless variables differs from that adopted in [3–7], since the presence of a magnetic field affects the trajectory geometry of particles with different masses. The dimensionless potential of an ideal focusing electrostatic system has the form [2]:

$$\begin{gathered}\phi(x,y,z)=z^2+f(x,y),\\ f_{xx}+f_{yy}+2=0.\end{gathered} \tag{5}$$

Due to the additive term z^2 in potential (5), the motion in the direction z in the field with potential (5) with the parameter $\beta=0$ is separated from the motion in the directions x and y and obeys the equation of harmonic oscillations:

$$\mu\ddot{z}=-2z, \tag{6}$$

whose solution has the form

$$z=z_0\cos\left(\sqrt{\frac{2}{\mu}}\tau\right)+\frac{\dot{z}_0}{\sqrt{2}}\sin\left(\sqrt{\frac{2}{\mu}}\tau\right). \tag{7}$$

Solution (7) guarantees ideal space time focusing of the beam in the planes $z=\pm z_0$ if charged particles start from the plane z_0 with any $\dot{z}_0$ or focusing of the beam in the plane $z=0$ at their start from different pointsz_0 with a zero z-component of the initial velocity.

Applying an external magnetic field transforms the equation of motion with respect to z:

$$\mu\ddot{z}=-2z+\beta(\dot{x}b_y-\dot{y}b_x). \tag{8}$$

Apparently, it is only at $\mathbf{b}=\mathbf{e}_z$ that Eq. (8) retains form (6), typical for electrostatics, guaranteeing that oscillations along the z-axis are independent of motion in the plane orthogonal to z. If $b_x\neq 0$ or $b_y\neq 0$, there is a ‘mixing’ of coordinates in system (3), and as a result the motion in the direction z is not separated, linearity is violated (unless the potential is a quadratic form and the system of equations of motion is not completely linear; see, for example, [13]). The field must be homogeneous, since the dependence of any component of the vector $\mathbf{b}$ on coordinates implies that (at least) one more component of the vector is different from zero and depends on the coordinates, by virtue of the equation div $\mathbf{b}=0$.

Thus, it is only possible to expand the class of fields with ideal space time focusing if systems of the following type are included in it:

electrostatic field of type (5) +
+ homogeneous magnetic field
directed along the z-axis.

Recall here that a homogeneous magnetic field provides ideal space time focusing of the beam in the following well-known cases, which we consider in dimensional coordinates assuming $\mathbf{B}=(0, 0, B)$.

1. $\mathbf{E}=0$, $\mathbf{B}\neq 0$. The period of rotation of particles in a homogeneous magnetic field

$$T=\frac{2\pi m}{qB} \tag{9}$$

does not depend on the initial data and is determined by the field strength and the charge/mass ratio.

Thus, the beam is ideally focused at the point coinciding with the starting point in the projection onto a plane orthogonal to the magnetic flux density vector, after a time equal to the period *T*. Particles of all masses are focused at this point but at different times. The motion along the vector **B** is uniform. Given a nonzero *z*-component of the initial particle velocity, the source becomes spatially separated from the detector (along the coordinate *z*). A time-of-flight spectrometer based on this principle is described in [14].

2. $\mathbf{E} \neq 0$, $\mathbf{B} \neq 0$, the vector **E** is parallel to the vector **B**. This case is similar to the first one, but the motion along *z* is uniformly accelerated, so the image size on the detector can be controlled.

3. $\mathbf{E} \neq 0$, $\mathbf{B} \neq 0$, the vector **E** is not parallel to the vector **B**. For definiteness, let the electric field vector be oriented as follows:

$$\mathbf{E} = (-E_x, 0, E_z).$$

This is a generalization of the case of crossed fields (known from elementary physics), which also provides ideal one-dimensional space-time focusing in a time multiple to the period *T* (see Eq. (9)), in a plane passing through the starting point orthogonally to the axis *x*. Unlike case 2, the focusing point is shifted relative to the starting point by a distance of $2\pi m E_x/(qB^2)$ in the direction *y*. Lines of the mass spectrum unfold as segments parallel to the vector **B**. The time-of-flight properties of a cycloidal mass spectrometer were used in patent [15].

A notetworthy combination of an electric field quadratic in one of the coordinates of the potential, and a homogeneous magnetic field directed along the same coordinate is the case of fields with axial symmetry. It was discussed in [16, 17] but in a different context.

Further consideration of the problem will be carried out only in dimensionless coordinates (2)–(4).

The most general form of potential (5) in the axisymmetric case is as follows:

$$\phi(r,z) = z^2 - \frac{r^2}{2} + \alpha \cdot \ln r, \quad \alpha \in \{-1, 0, 1\}. \tag{10}$$

This expression includes the field of the orbital trap [1] ($\alpha = 1$), confining particles at certain initial velocities, as well as the field of the hyperboloid ($\alpha = 0$) and the field of the quadro-logarithmic type ($\alpha = -1$), which fundamentally do not confine the particles along the coordinate *r* in the absence of a magnetic field. Potential (10) provides harmonic oscillations of the ion in the direction *z* (guaranteeing ISTF) and its radial-azimuthal motion in the field of the effective potential, which has the form

$$U_0(r) = \mu \frac{r_0^4 \dot{\gamma}_0^2}{2r^2} - \frac{r^2}{2} + \alpha \ln r. \tag{11}$$

The dependence of U_0 on the mass μ generated in this case of non-dimensionalization disappears if we take into account that the total energy of the particle is related to the initial data by the relation

$$\mu\left(\frac{\dot{r}_0^2}{2} + \frac{r_0^2 \dot{\gamma}_0^2}{2} + \frac{\dot{z}_0^2}{2}\right) = E = E_r + E_\gamma + E_z. \tag{12}$$

Then $\mu \frac{r_0^2 \dot{\gamma}_0^2}{2} = E_\gamma$ and

$$U_0(r) = E_\gamma \frac{r_0^2}{r^2} - \frac{r^2}{2} + \alpha \ln r. \tag{13}$$

As a result of immersing system (10) in a homogeneous magnetic field, effective potential (11) is replaced by the potential

$$U_\beta(r) = \left(\dot{\gamma}_0 + \frac{\beta}{2\mu}\right)^2 \frac{\mu r_0^4}{2r^2} + \left(\frac{\beta^2}{4\mu} - 1\right)\frac{r^2}{2} + \alpha \ln r \tag{14}$$

and, accordingly, deformation of the initial radial-azimuthal motion occurs if ideal spatio-temporal focusing of the beam along the coordinate *z* is maintained. The boundaries of the region of radial motion substantially depend on the mass here.

The well-known Penning trap corresponds to the zero value of the parameter α (see, for example, monograph [18]). This scenario is sufficiently well-understood, so let us focus on the cases when $\alpha = 1\pm$.

Denoting

$$a = \left(\dot{\gamma}_0 + \frac{\beta}{2\mu}\right)^2 \frac{\mu r_0^4}{2}, c = \left(\frac{\beta^2}{4\mu} - 1\right), \tag{15}$$

we write expression (14) in the form

$$U_\beta(r) = \frac{a}{r^2} + \frac{cr^2}{2} + \alpha \ln r. \qquad (16)$$

Since $a > 0$, $U_\beta(r) \to +\infty$ for $r \to 0$. The behavior of $U_\beta(r)$ at $r \to +\infty$ is determined by the sign of the parameter c. For $c > 0$, $U_\beta(r) \to +\infty$ at $r \to +\infty$, and the dependence $U_\beta(r)$ has a minimum, forming a potential well trapping ions. For $c < 0$, $U_\beta(r) \to -\infty$ at $r \to +\infty$ and the effective potential well exists only in the presence of both a minimum and a maximum of $U_\beta(r)$; an ion can only be trapped in the well if certain conditions relating the initial data of the ion to the field parameters similar to the case of electrostatics are met [19].

Thus, the equation $\partial U_\beta(r)/\partial r = 0$ has two real roots for the case when $c < 0$:

$$r^2 = \frac{-\alpha \pm \sqrt{\alpha^2 + 8ac}}{2c}$$

provided that the discriminant is positive, namely, at $8ac > -\alpha^2$.

Then,

$$-\alpha^2/(8a) < c < 0. \qquad (17)$$

The condition $r^2 > 0$ leads to the inequality

$$-\alpha \pm \sqrt{\alpha^2 + 8ac} < 0,$$

which, if relation (17) is satisfied, always holds for $\alpha = 1$ and never for $\alpha = -1$. Using expressions (15), (17) for $\alpha = 1$, we obtain the inequalities

$$-1 < 4\mu r_0^4 \left(\frac{\beta^2}{4\mu} - 1\right) \cdot \left(\dot{\gamma}_0 + \frac{\beta}{2\mu}\right)^2 < 0. \qquad (18)$$

The requirement that the middle part of expression (18) (that is, the right-hand side of double inequality (18)) be negative is ensured by fulfilling the condition $\beta^2 < 4\mu$, which is always true at zero magnetic field. The left-hand side of double inequality (18) is fulfilled by choosing the corresponding initial data.

The equality $c = 0$ for $\alpha = 1$ makes the existence of the effective potential well (16) possible for any initial data. If $c > 0$, the well is guaranteed for ions with arbitrary starting conditions at any value of α. Thus, the variant $\alpha = 1$ corresponds to a system capable of trapping ions with any masses μ, but only for certain initial data if $\mu \geq \beta^2/4$.

If the condition $c > 0$ is fulfilled, a system with $\alpha = -1$ can confine ions, which is impossible in a purely electric field. In this case, the requirement $\mu < \beta^2/4$, equivalent to the inequality $c > 0$, sets the threshold value of the mass of ions that are radially stable in a magnetic field with the strength β. A device based on such fields can also find applications in mass spectrometry.

In conclusion, let us point out the case of a weakly varying homogeneous magnetic field mentioned in [13], which also expands the given class of fields with ISTF.

Conclusion

It can be concluded from the study that a necessary condition for ideal space-time focusing in the system in at least one directions is either linearity of the equation with separated motion in this direction, or, given mixed coordinates, linearity of the entire system of motion equations (in particular, if the frequency multiplicity of the oscillations providing focusing in several directions is satisfied). The magnetic field must only be homogeneous; there are potential applications for its effect on ions as a holding field, including as a compensator for the repulsive forces of the electric field.

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