CHAINS OF FUNDAMENTAL MUTUALLY HOMOGENEOUS FUNCTIONS WITH A COMMON REAL EIGENVALUE

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This work continues our studies in properties of the mutually homogeneous functions (MHF) being a generalization of Euler homogeneous functions. MHF can be used in the synthesis of electric and magnetic fields for electron systems and ion-optical ones with special properties. A chain of functions corresponding to multiple real eigenvalues of the matrix of basic functional relations for MHF has been considered. Functional relations answering such functions were derived. General formulas for the solutions of the obtained functional relations were derived. The obtained functions were shown to be a refinement of the associated homogeneous functions introduced by Gel'fand. Typical differential and integral properties of the obtained functions were investigated, and a generalization of the Euler theorem was proved (Euler criterion) for differentiable functions.

Keywords: functional equation, homogeneous function, associated homogeneous function, mutually homogeneous functions

ЦЕПОЧКИ ФУНДАМЕНТАЛЬНЫХ ВЗАИМНО-ОДНОРОДНЫХ ФУНКЦИЙ С ОБЩИМ ВЕЩЕСТВЕННЫМ СОБСТВЕННЫМ ЧИСЛОМ

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Данная работа продолжает изучение свойств взаимно-однородных функций (ВОФ), которые являются обобщением функций, однородных по Эйлеру; ВОФ могут использоваться при синтезе электрических и магнитных полей электронно- и ионно-оптических систем со специальными свойствами. Рассматривается цепочка функций, соответствующая кратным вещественным собственным значениям матрицы базовых функциональных уравнений для ВОФ. Выведены функциональные соотношения, характеризующие такие функции, а также общие формулы для функций, являющихся решениями полученных функциональных соотношений. Показано, что полученный класс функций представляет собой уточнение присоединенных однородных функций Гельфанда. Исследованы типичные дифференциальные и интегральные свойства полученного класса функций, а для дифференцируемых функций доказано обобщение теоремы Эйлера (критерий Эйлера).

Ключевые слова: функциональное уравнение, однородная функция, присоединенная однородная функция, взаимно-однородные функции

Introduction

This paper continues a series of studies [1−4] considering the properties of homogeneous harmonic functions and their applications for synthesis of electric and magnetic fields for electron and ion-optical systems with special properties [5−8]. Carrying on where [9] left off, our work heavily relies on the results presented therein.

A function $f(x_1, x_2, ..., x_n)$ is called Euler-homogeneous with the degree of homogeneity equal to p if the identity

$$f(\lambda x_1, \lambda x_2, ..., \lambda x_n) = \lambda^p f(x_1, x_2, ..., x_n). \quad (1)$$

holds true for any real values of λ.

The main properties and theorems on Euler-homogeneous functions are described in monograph [10]. In particular, any homogeneous function of degree p can be represented as

$$f(x_1, x_2, ..., x_n) = \\ = x_1^p \, h(x_2/x_1, x_3/x_1, ..., x_n/x_1), \quad (2)$$

where $h(t_2, t_3, ..., t_n)$ is a certain function of $(n - 1)$ variables, while any function taking the form (2) is homogeneous with the degree p.

The function $f(x_1, x_2, ..., x_n)$ is called positively homogeneous in Euler terms with the degree p if identity (1) holds true for any positive real values of λ, while the identity is not guaranteed to hold for negative real values of λ (for example, function $f(x) \neq x$). Imposing a constraint that $\lambda > 0$, in particular, allows to safely operate random real degrees of homogeneity in Eq. (1): additional steps need to be taken for a random real degree p to determine the power function λ^p at negative values of λ to satisfy the condition

$$\lambda_1^p \lambda_2^p = (\lambda_1 \lambda_2)^p.$$

Positively homogeneous function $f(x_1, x_2, ..., x_n)$ of degree p can be represented in the form:

$$\text{if } x_1 > 0 \; f(x_1, x_2, ..., x_n) = \\ = x_1^p \, h(x_2/x_1, x_3/x_1, ..., x_n/x_1); \quad (3)$$

$$\text{if } x_1 < 0 \; f(x_1, x_2, ..., x_n) = \\ = (-x_1)^p \, g(x_2/x_1, x_3/x_1, ..., x_n/x_1), \quad (4)$$

where $h(t_2, t_3, ..., t_n)$ and $g(t_2, t_3, ..., t_n)$ are functions of $(n - 1)$ variables independent of each other (in general).

Eqs. (3) and (4) are obtained from relation (1) by substituting into it the values $\lambda = +1/x_1$ for $x_1 > 0$ and $-1/x_1$ for $x_1 < 0$, if the functions $h(t_2, t_3, ..., t_n)$ and $g(t_2, t_3, ..., t_n)$ are defined as follows:

$$h(t_2, t_3, ..., t_n) = f(+1, t_2, t_3, ..., t_n),$$

$$g(t_2, t_3, ..., t_n) = f(-1, -t_2, -t_3, ..., -t_n).$$

If $x_1 = 0$, the function $f(0, x_2, x_3, ..., x_n)$ is positively Euler-homogeneous of degree p with less variables, so parametrization of the form (3), (4) can be applied to it. A recursive process of constructing complete parametrization for a positively homogeneous function $f(x_1, x_2, ..., x_n)$ stops when a set of variables $x_1, x_2, ..., x_n$ is exhausted.

Consider the functions taking the form

$$\text{if } x_1 > 0: f_{p,k}(x_1, x_2, ..., x_n) = \\ = (1/k!) \, x_1^p \, (q \ln x_1)^k \times \quad (5) \\ \times h(x_2/x_1, x_3/x_1, ..., x_n/x_1),$$

$$\text{if } x_1 < 0: f_{p,k}(x_1, x_2, ..., x_n) = \\ = (1/k!) \, (-x_1)^p \, (q \ln (-x_1))^k \times \quad (6) \\ \times g(x_2/x_1, x_3/x_1, ..., x_n/x_1),$$

where p, q are real constants; k is an integer index ($k = 0, 1, 2, ...$); $h(t_2, t_3, ..., t_n)$, $g(t_2, t_3, ..., t_n)$ are certain functions of $(n - 1)$ variables; the values of the variable x_1 satisfy the condition $x_1 \neq 0$.

Given the functional relations

$$f_i(\lambda x_1, \lambda x_2, ..., \lambda x_n) = \\ = \sum a_{ij}(\lambda) f_j(x_1, x_2, ..., x_n), \quad (7)$$

where $i, j = 1, 2, ..., k$, and the functions $a_{ij}(\lambda)$ are unknown in advance, then, in a particular case when all eigenvalues of the matrix $\|a_{ij}(\lambda)\|$ are real numbers p equal to each other (see [9]), functions taking the form (5), (6) may qualify as possible solutions to functional equations of the form (7).

Using direct substitution, we can confirm that for $\forall \lambda > 0$, functions (5), (6) satisfy the functional relations

$$f_{p,k}(\lambda x_1, \lambda x_2, \ldots, \lambda x_n) =$$
$$= \sum_{j=0,k} a_{k-j}(\lambda) f_{p,j}(x_1, x_2, \ldots, x_n), \qquad (8)$$

where $a_j(\lambda) = (1/j!) \lambda^p (q \ln \lambda)^j$.

The objectives of this study are, firstly, developing general formulae for the functions satisfying functional equations (8) provided that functions $a_j(\lambda)$ take the form

$$a_j(\lambda) = (1/j!) \lambda^p (q \ln \lambda)^j,$$

and, secondly, proving certain important theorems on the obtained class of real functions of multiple variables.

Relationship of functions (5) with associated homogeneous Gel'fand functions

Functions (5) and (6) satisfying functional relations (8) are a refinement of associated homogeneous Gel'fand functions as defined in [11, 12]. However, these studies falsely assume that the system of functional relations (8) is a bidiagonal matrix with functions $a(\lambda)$ for the main diagonal and $b(\lambda)$ for the auxiliary one, unknown in advance. Unfortunately, while this insignificant mistake in the formal definition did not affect the other fundamental results obtained in [11, 12], was further uncritically disseminated in subsequent publications by other authors [13−20]. We could only find mentions of this inaccuracy in [21, 22] but even in these instances the authors omitted the factor $1/k!$ in the respective formulae from consideration. This shortcoming is absent in earlier formulae presented in [23]. Moreover, no analysis of the *general* solution was performed in [21−23] for the obtained functional equations after verifying the required functional relations for the given functions (i.e., after obtaining a particular solution).

It is easy to prove at least for differentiable functions[1] that a bidiagonal system of functional equations (8) can have nondegenerate solutions different from the null equation only when $a(\lambda) = \lambda^p$ and $b(\lambda) = \lambda^p(q \ln \lambda)$. At the same time, these solutions (if they exist) must have a form of linear combinations with constant factors composed of functions (7) [21]. Unfortunately,

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[1] For this conclusion, it is actually sufficient to impose that each of the functions $a(\lambda)$ and $b(\lambda)$ is continuous in at least one point of $\lambda > 0$. Rigorous proof of this statement is not complicated but lies beyond the scope of our study.

even when $k = 3$, functions (7) do not satisfy the system of bidiagonal relations (8), and we can prove that these functional relations have essentially no solutions for $k \geq 3$ [21].

One of the goals of this work is to reintroduce mathematical rigor to associated homogeneous Gel'fand functions, as well as to study some interesting properties of the obtained functions.

We should stress that we consider a rather narrow subclass of functions that is the closest to associated homogeneous Gel'fand functions. The general solution for functional equations (8) with the functions $a_j(\lambda)$ unknown in advance is far more extensive, and we are in fact planning another publication on this subject.

General formulae

So as not to confuse the constructions we consider with the associated homogeneous functions in terms of Gel'fand definitions [11, 12], let us add the following definition.

Definition. *A semi-infinite chain of functions* $f_{p,k}(x_1, x_2, \ldots, x_n)$, *where* $k = 0, 1, 2, \ldots$, *and the functions* $f_{p,k}(x_1, x_2, \ldots, x_n)$ *satisfy the functional relations*

$$f_{p,k}(\lambda x_1, \lambda x_2, \ldots, \lambda x_n) =$$
$$= \sum_{j=0,k} (1/(k-j)!) \lambda^p (q \ln \lambda)^{k-j} \times \qquad (9)$$
$$\times f_{p,j}(x_1, x_2, \ldots, x_n),$$

for all $\lambda > 0$ *is called fundamental associated homogeneous functions of degree p and order k.*

Changing the order of summation, relations (9) can be written in an equivalent form as

$$f_{p,k}(\lambda x_1, \lambda x_2, \ldots, \lambda x_n) =$$
$$= \sum_{j=0,k} (1/j!) \lambda^p (q \ln \lambda)^j \times$$
$$\times f_{p,k-j}(x_1, x_2, \ldots, x_n).$$

Parameter q is responsible for normalization of the fundamental associated homogeneous functions and does not affect the rest of their properties. After substituting

$$f_{p,j}(x_1, x_2, \ldots, x_n) =$$
$$= q^j F_{p,j}(x_1, x_2, \ldots, x_n),$$

the parameter q is reduced in functional relations (9), and functions $F_{p,j}(x_1, x_2, \ldots, x_n)$ take the meaning of normalized fundamental associated homogeneous functions corresponding to the choice $q = 1$.

We need to find the general formulae for the functions satisfying functional relations (9), similar to formulae (3) and (4). The solution is provided by the following theorem.

Theorem 1. *Chain of fundamental associated homogeneous functions $f_{p,k}(x_1, x_2, ..., x_n)$ of degree p and order k, obeying functional relations (9) for all $\forall \lambda > 0$, has the following one-to-one representation for $x_1 \neq 0$:*

$$\text{if } x_1 > 0 \ f_{p,k}(x_1, x_2, ..., x_n) =$$

$$= \Sigma_{j=0,k} \ (1/(k-j)!) \ x_1^p \ (q \ln x_1)^{k-j} \times \qquad (10)$$

$$\times h_j(x_2/x_1, x_3/x_1, ..., x_n/x_1);$$
$$\text{if } x_1 < 0 \ f_{p,k}(x_1, x_2, ..., x_n) =$$

$$= \Sigma_{j=0,k} \ (1/(k-j)!) \ (-x_1)$$

$$^p \ (q \ln (-x_1))^{k-j} \times \qquad (11)$$

$$\times g_j(x_2/x_1, x_3/x_1, ..., x_n/x_1),$$

where $g_j(t_2, t_3, ..., t_n)$, $h_j(t_2, t_3, ..., t_n)$ are real functions of $(n-1)$ variables, which have a one-to-one correspondence with functions $f_{p,k}(x_1, x_2, ..., x_n)$.

The reverse is also true: a chain of functions given by Eqs. (10) and (11) obeys functional relations (9) for $x_1 \neq 0$ with randomly chosen functions $g_j(t_2, t_3, ..., t_n)$ and $h_j(t_2, t_3, ..., t_n)$.

When $x_1 = 0$ and $x_2 \neq 0$, parametrization for fundamental associated homogeneous functions $f_{p,k}(0, x_2, x_3, ..., x_n)$ of degree p and order k is constructed similar to Eqs. (10), (11). Complete parametrization for functions $f_{p,k}(x_1, x_2, ..., x_n)$ is repeated recursively until the set of variables $x_1, x_2, ..., x_n$ is exhausted.

Proof. Let us confine ourselves to considering only the case when $x_1 > 0$, since the case when $x_1 < 0$ is derived from it by substituting $x_1 = -x_1$, with relations (9) remaining unchanged.

When $k = 0$, relations (9) transform into homogeneity relation (1), while the function $f_{p,0}(x_1, x_2, ..., x_n)$ turns out to be a positively homogeneous function of degree p which is defined at $x_1 > 0$. Consequently, Eq. (10) holds true with $k = 0$, as it coincides with Eq. (3) for positively homogeneous functions, while the function $h_0(t_2, t_3, ..., t_n)$ is mapped one-to-one using the obtained function $f_{p,0}(x_1, x_2, ..., x_n)$.

Let us employ tmathematical induction.

Suppose Eqs. (10) are proved for all values of k satisfying the inequality $0 \leq k < m$. Let us write the function $f_{p,m}(x_1, x_2, ..., x_n)$ for $x_1 > 0$ as

$$f_{p,m}(x_1, x_2, ..., x_n) = x_1^p$$

$$H(x_1, x_2, ..., x_n) +$$

$$+ \Sigma_{j=0,m-1} \ (1/(m-j)!) \ x_1^p \ (q \ln x_1)^{m-j} \times \qquad (12)$$

$$\times h_j(x_2/x_1, x_3/x_1, ..., x_n/x_1),$$

where the functions $h_j(t_2, t_3, ..., t_n)$ for $j = 0, 1, m-1$ have been already defined at the previous steps of the proof. It is required to find the form that the function $H(x_1, x_2, ..., x_n)$, which has a meaning at $x_1 > 0$, should take for the identity

$$f_{p,m}(\lambda x_1, \lambda x_2, ..., \lambda x_n) -$$

$$- \lambda^p f_{p,m}(x_1, x_2, ..., x_n) -$$

$$- \Sigma_{k=0,m-1} \ (1/(m-k)!) \ \lambda^p \ (q \ln \lambda)^{m-k} \times \qquad (13)$$

$$\times f_{p,k}(x_1, x_2, ..., x_n) = 0.$$

to hold true for $\forall \lambda > 0$.

After simplifying expression (13) given that the functions $f_{p,k}(x_1, x_2, ..., x_n)$ can be replaced by relations (10) for $0 \leq k < m$, we obtain the condition:

$$\text{if } \forall \lambda > 0, x_1 > 0$$

$$H(\lambda x_1, \lambda x_2, ..., \lambda x_n) = H(x_1, x_2, ..., x_n).$$

Consequently, the function $H(x_1, x_2, ..., x_n)$ must be a positively homogeneous function of zero degree, defined for $x_1 > 0$. This condition is necessary and sufficient to fulfil equality (13), because all algebraic transformations simplifying expression (13) are reversible.

According to Eq. (3), when $x_1 > 0$, the function $H(x_1, x_2, ..., x_n)$ can be represented as

$$H(x_1, x_2, ..., x_n) =$$

$$= h_m(x_2/x_1, x_3/x_1, ..., x_n/x_1),$$

where $h_m(t_2, t_3, ..., t_n)$ is a certain new function of $(n-1)$ variables.

Next, if we substitute the values $x_1 = 1$ into equality (12), we obtain the condition

$$f_{p,m} \ (1, x_2, ..., x_n) = h_m(x_2, x_3, ..., x_n),$$

which implies a one-to-one correspondence between the functions $f_{p,m}$ and h_m.

Thus, with $x_1 > 0$, Eq. (10) holds true for $k = m$ as well.

Theorem 1 is proved.

The chain of associated homogeneous functions can be also represented in parameterized form by other means. For example, a method for constructing the most generalized type of parametrization can be formulated as the following theorem.

Theorem 2. *Suppose $\omega_p(x_1, x_2, ..., x_n)$ is a positively homogeneous function of degree p, $\psi_q(x_1, x_2, ..., x_n)$ is a positively homogeneous function of degree $q \neq 0$, and $\psi_2(x_1, x_2, ..., x_n)$, $\psi_3(x_1, x_2, ..., x_n), ..., \psi_n(x_1, x_2, ..., x_n)$ are positively homogeneous functions of zero degree.*

Let these functions be defined at any point of the domain Ω. *Additionally, suppose that the function* ω_p *does not become zero in the domain* Ω, *the function* ψ_q *is strictly positive, the functions* $\psi_2, \psi_3, ..., \psi_n$ *are functionally independent.*

Then the fundamental associated homogeneous functions $f_{p,k}(x_1, x_2, ..., x_n)$, *which obey the functional relations (9) for* $\forall \lambda > 0$, *can be mapped one-to-one as follows in the domain* Ω:

$$f_{p,k}(\boldsymbol{x}) =$$
$$= \Sigma_{j=0,k} \, (1/(k-j)!) \, \omega_p(\boldsymbol{x}) \, (\ln \psi_q(\boldsymbol{x}))^{k-j} \times \quad (14)$$
$$\times \, h_j(\psi_2(\boldsymbol{x}), \psi_3(\boldsymbol{x}), ..., \psi_n(\boldsymbol{x})),$$

where $\boldsymbol{x} = (x_1, x_2, ..., x_n)$, and $h_j(t_2, t_3, ..., t_n)$ are certain real functions of $(n-1)$ variables.

P r o o f. When $k = 0$, the function $f_{p,0}(x_1, x_2, ..., x_n)$ is a positively homogeneous function of degree p, while the function

$$f_{p,0}(x_1, x_2, ..., x_n)/\omega_p(x_1, x_2, ..., x_n)$$

is a well-defined positively homogeneous function of zero degree. It can be represented as $h_0(\psi_2, \psi_3, ..., \psi_n)$, as it is functionally dependent on the functionally independent functions ψ_2, $\psi_3, ..., \psi_n$. Indeed, if we can find such a positively homogeneous function $\psi(x_1, x_2, ..., x_n)$ of zero degree, which forms a functionally independent set with the functions

$$\psi_2(x_1, x_2, ..., x_n),$$

$$\psi_3(x_1, x_2, ..., x_n), ..., \psi_n(x_1, x_2, ..., x_n),$$

then the free variables $x_1, x_2, ..., x_n$ can be expressed in terms of functionally independent positively homogeneous functions $\psi, \psi_2, ..., \psi_n$ of zero degree. Then any function of the variables $x_1, x_2, ..., x_n$ would be a positively homogeneous functions of zero degree. This cannot be true, so the corresponding function $h_0(\psi_2, \psi_3, ..., \psi_n)$ must exist, and thus, Eq. (14) is fulfilled for $k = 0$. Further proof by induction repeats the proof of Theorem 1 practically verbatim.

Theorem 2 is proved.

Using Eqs. (14), the entire space R^n is divided into non-intersecting conic[2] domains Ω_s, for each of which the selected functions $\omega_p(x_1, x_2, ..., x_n)$ and $\psi_q(x_1, x_2, ..., x_n)$ do not become zero[3], and the functions

$$\psi_2(x_1, x_2, ..., x_n),$$

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[2] The domain Ω is called a hypercone, if it follows from the condition $\boldsymbol{x} \in \Omega$ that the condition $\lambda\boldsymbol{x} \in \Omega$ is also satisfied for any points $\lambda\boldsymbol{x}$ at random values $\lambda > 0$.

[3] If the function ψ_q is negative in the given domain, it is replaced by $-\psi_q$.

$$\psi_3(x_1, x_2, ..., x_n), ..., \psi_n(x_1, x_2, ..., x_n)$$

form a functionally independent set of positively homogeneous functions of zero degree. Generally speaking, we construct parametrization (14) for each of the domains Ω_s using a separate set of functions $h_j(t_2, t_3, ..., t_n)$ unrelated to the functions $h_j(t_2, t_3, ..., t_n)$ used for other domains. The boundaries between the conic domains are conic surfaces of smaller dimensions, along which the given functions $f_{p,k}(x_1, x_2, ..., x_n)$ act as fundamental associated homogeneous functions of smaller dimensions, with parametrization constructed by a similar algorithm.

Importantly, parametrization of fundamental associated homogeneous functions $f_{p,k}(x_1, x_2, ..., x_n)$ is partitioned into several independent branches as a result; moreover, such a partition depends on the selected auxiliary functions $\omega_p(x_1, x_2, ..., x_n)$ and $\psi_q(x_1, x_2, ..., x_n)$, and, to a lesser degree, on the functions

$$\psi_2(x_1, x_2, ..., x_n),$$

$$\psi_2(x_1, x_2, ..., x_n), ..., \psi_n(x_1, x_2, ..., x_n),$$

and does not reflect the inner structure of the chain of functions parameterized.

Partitioning the space R^n into several independent branches can be avoided as the following theorem implies.

Theorem 3. *A chain of fundamental associated homogeneous functions* $f_{p,k}(x_1, x_2, ..., x_n)$, *which adheres to functional relations (9) for all* $\forall \lambda > 0$ *can be mapped one-to-one as follows:*

$$f_{p,k}(\boldsymbol{x}) = \Sigma_{j=0,k} \, (1/(k-j)!) \, r^p \, (q \ln r)^{k-j} \times$$
$$\times \, h_j(x_1/r, x_2/r, ..., x_n/r), \quad (15)$$

where $\boldsymbol{x} = (x_1, x_2, ..., x_n)$, $r = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$ *and* $h_j(t_1, t_2, ..., t_n)$ *are arbitrary real functions given on the surface of a unit hypersphere*

$$t_1^2 + t_2^2 + ... + t_n^2 = 1,$$

with a one-to-one correspondence to the functions $f_{p,k}(x_1, x_2, ..., x_n)$.

P r o o f. If $k = 0$, we can establish that Eq. (15) holds true for a positively homogeneous function $f_{p,0}(x_1, x_2, ..., x_n)$ after substituting $\lambda = 1/r$ into homogeneity relation (1) and using the function $h_0(t_1, t_2, ..., t_n) = f_{p,0}(t_1, t_2, ..., t_n)$ (recall that each of the functions $h_j(t_1, t_2, ..., t_n)$ is defined only for the surface of a unit hypersphere $t_1^2 + t_2^2 + ... + t_n^2 = 1$). Further proof by induction repeats the proof of Theorem 1 practically verbatim.

Theorem 3 is proved.

Relations (9) imply that the linear combination with constant coefficients comprised from several chains of fundamental associated homogeneous functions of degree p and order k is also a chain of fundamental associated homogeneous functions of degree p and order k. Besides, if $f_{p,k}(x_1, x_2, ..., x_n)$ is a chain of fundamental associated homogeneous functions of degree p and order k, then a new chain of functions

$$g_{p,k}(x_1, x_2, ..., x_n) = f_{p,k-1}(x_1, x_2, ..., x_n)$$

with an index shift, supplemented by a leading zero $g_{p,0}(x_1, x_2, ..., x_n) = 0$, is also a chain of fundamental associated homogeneous functions of degree p and order k.

Eqs. (10), (11) (as well as (14) or (15)) illustrate the validity of Gel'fand's hypothesis that any chains of associated homogeneous functions of degree p and order k are obtained from the main chains with a nonzero first term by shifting the index k and subsequent summation. At the same time, all elements of the main chain of functions are reconstructed one-to-one by to its first term according to a certain rule; the accurate formulation of this rule reflects the researcher's preferences and, generally speaking, can be different for the same initial function. In case of the theorems proved above, the respective chains of fundamental associated homogeneous functions have the form

a) for Eqs. (10), (11):

$$\text{if } x_1 > 0 \; f^{(j)}_{p,k}(x) =$$

$$= (x_1^p / k!) \, (q \ln x_1)^k \times$$

$$\times h_j(x_2/x_1, x_3/x_1, ..., x_n/x_1);$$

$$\text{if } x_1 < 0 : f^{(j)}_{p,k}(x) =$$

$$= ((-x_1)^p / k!) \, (q \ln (-x_1))^k \times$$

$$\times g_j(x_2/x_1, x_3/x_1, ..., x_n/x_1);$$

b) for Eq. (14):

$$f^{(j)}_{p,k}(x) = \omega_p(x) / k! \, (\ln \psi_q(x))^k \times$$

$$\times h_j(\psi_2(x), \psi_3(x), ..., \psi_n(x));$$

c) for Eq. (15):

$$f^{(j)}_{p,k}(x) =$$

$$= (r^p / k!) \, (q \ln r)^k \, h_j(x_1/r, x_2/r, ..., x_n/r).$$

Remark. As follows from Eqs. (14), the fundamental associated homogeneous functions are actually linear combinations of chains of functions taking the form

$$(1/k!) \, R_p(x_1, x_2, ..., x_n) \times$$

$$\times (\ln S_q(x_1, x_2, ..., x_n))^k,$$

where $R_p(x_1, x_2, ..., x_n)$ are random positively homogeneous functions of degree p, and $S_q(x_1, x_2, ..., x_n)$ are fixed positively homogeneous functions of degree q, for which we also shift the index k and supplement the shifted chains with leading zeros.

The situation will not change and no new functions can be obtained if we demand that the functions $S_q(x_1, x_2, ..., x_n)$ are random positively homogeneous functions of degree q.

In particular, this approach allows to formulate the fundamental associated homogeneous functions more elegantly without using artificially derived variables x_1. Changing the selected function $S_q(x_1, x_2, ..., x_n)$ makes the current main chains secondary, and, vice versa, the chains that were previously secondary the main ones. Because of this, the definition of the main chains of fundamental associated homogeneous functions is fairly arbitrary and depends on the selected parametrization of fundamental associated homogeneous functions.

Differentiation and integration of associated homogeneous functions

If an Euler-homogeneous function $f(x_1, x_2, ..., x_n)$ of degree p is differentiable, then its derivatives with respect to the variables $x_1, x_2, ..., x_n$ are homogeneous functions of degree $(p - 1)$ [10]. A similar statement is valid for the associated homogeneous functions. Let us formulate and prove the following theorem.

Theorem 4 (on differentiation). *If $f_{p,k}(x_1, x_2, ..., x_n)$ is a chain of fundamental associated homogeneous functions of degree p and order k, and the functions $f_{p,k}(x_1, x_2, ..., x_n)$ are differentiable, then their first partial derivatives $\partial f_{p,k}/\partial x_i$ with respect to the variables $x_1, x_2, ..., x_n$ form chains of fundamental associated homogeneous functions of degree $(p - 1)$ and order k.*

P r o o f. The statement of the theorem follows from a term-by-term differentiation of the right and the left-hand sides of Eq. (9) with respect to the variable x_i.

Theorem 4 is proved.

A similar statement is valid for integration.

Theorem 5 (on integration). *If $f_{p,k}(x_1, x_2, ..., x_n)$ is a chain of fundamental associated homogeneous functions of degree p and order k, then integrals represented as*

$$F_{p,k}(x_1, x_2, ..., x_n) =$$

$$= \int_0^{x_i} f_{p,k}(x_1, x_2, ..., x_{i-1}, t, x_{i+1}, ..., x_n) dt$$

(if they exist) form a chain of fundamental associated homogeneous functions of degree (p + 1) and order k.

Significantly, the initial point of integration is zero.

P r o o f. The proof follows from term-by-term differentiation with respect to the variable t in the interval $t \in [0, x_i]$ of relation (8) after substituting $x_i \to t$ in it in view of the equality

$$\int_0^{x_i} f_{p,k}(\lambda x_1, \lambda x_2, ..., \lambda x_{i-1}, \lambda t, \lambda x_{i+1}, ..., \lambda x_n) dt =$$

$$= \frac{1}{\lambda} \int_0^{\lambda x_i} f_{p,k}(\lambda x_1, \lambda x_2, ..., \lambda x_{i-1}, \tau, \lambda x_{i+1}, ..., \lambda x_n) d\tau.$$

Theorem 5 is proved.

It is also possible to consider the integrals

$$F_{p,k}(x_1, ..., x_n) =$$

$$= \int_{a_k}^{x_i} f_{p,k}(x_1, ..., x_{i-1}, t, x_{i+1}, ..., x_n) dt +$$

$$+ g_k(x_1, ..., x_{i-1}, x_{i+1}, ..., x_n),$$

where the functions $g_k(x_1, x_2, ..., x_{i-1}, x_{i+1}, ..., x_n)$ are such that the obtained functions $f_{p,k}(x_1, x_2, ..., x_n)$ form a chain of fundamental associated homogeneous functions of degree (p + 1) and order k. We can prove that such functions g_k do indeed exist and can be expressed in terms of the functions $f_{p,k}(x_1, x_2, ..., x_n)$ with a one-to-one correspondence up to the additive elements in the form of fundamental associated homogeneous functions of degree (p + 1) and order k depending on the variables $x_1, x_2, ..., x_{i-1}, x_{i+1}, ..., x_n$. The proof of this statement is given in the following section.

Theorem 6 (on fractional differentiation). *If $f_{p,k}(x_1, x_2, ..., x_n)$ is a chain of fundamental associated homogeneous functions of degree p and order k, then their fractional derivatives $F_{p,k}(x_1, x_2, ..., x_n)$ of order $\alpha \in (0, 1)$ (Riemann−Liouville integrals of order α [24−26]), expressed as*

$$F_{p,k}(x_1, x_2 ..., x_n) =$$

$$= \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dx_i^m} \int_0^{x_i} (x_i - t)^{m-\alpha-1} \times$$

$$\times f_{p,k}(x_1, x_2 ..., x_{i-1}, t, x_{i+1}, ..., x_n) dt,$$

form a chain of fundamental associated homogeneous functions of degree (p − α) and order k (if such integrals exist, in particular, if m − α > 0).

Significantly, the initial point of integration is zero.

P r o o f. The proof follows from term-by-term application of the linear convolution operator $L[f]$ to relation (8):

$$L\left[f(x_1, x_2 ..., x_n) \right] =$$

$$= \int_0^{x_i} (x_i - t)^{m-\alpha-1} \times$$

$$\times f(x_1, x_2 ..., x_{i-1}, t, x_{i+1}, ..., x_n) dt,$$

where we should also take into account the equality

$$\int_0^{x_i} (x_i - t)^{m-\alpha-1} \times$$

$$\times f_{p,k}(\lambda x_1, \lambda x_2 ..., \lambda x_{i-1}, \lambda t, \lambda x_{i+1}, ..., \lambda x_n) dt =$$

$$= \frac{1}{\lambda^{m-\alpha}} \int_0^{\lambda x_i} (\lambda x_i - \tau)^{m-\alpha-1} \times$$

$$\times f_{p,k}(\lambda x_1, \lambda x_2 ..., \lambda x_{i-1}, \tau, \lambda x_{i+1}, ..., \lambda x_n) d\tau.$$

As a result, we obtain a chain of fundamental associated homogeneous functions of degree (m + p − α) and order k, which, after m-fold differentiation with respect to the variable x_i, becomes a chain of fundamental associated homogeneous functions of degree (p − α) and order k.

Theorem 6 is proved.

Theorem 7 (on convolution with a generalized Abel kernel). *If $f_{p,k}(x_1, x_2, ..., x_n)$ is a chain of fundamental associated homogeneous functions of degree p and order k, then provided that there the corresponding integrals exist, their convolution with the generalized Abel kernel expressed as*

$$F_{p,k}(x_1, x_2 ..., x_n) =$$

$$= \int_0^{x_1} \cdots \int_0^{x_n} \left(x_1^{k_1} - t_1^{k_1} \right)^{\frac{\mu_1-1}{k_1}} \cdots \left(x_n^{k_n} - t_n^{k_n} \right)^{\frac{\mu_n-1}{k_n}} \times$$

$$\times f_{p,k}(t_1, t_2 ..., t_n) dt_1 ... dt_n,$$

where $\forall \mu_i > 0$, *forms a chain of fundamental associated homogeneous functions of degree* $p + \mu_1 + \ldots + \mu_n$ *and order k. The result for partial convolution with respect to the variables* x_1, x_2, $\ldots$, x_m *is a chain of fundamental associated homogeneous functions of degree* $p + \mu_1 + \ldots + \mu_m$ *and order k.*

Significantly, the initial point of integration is zero.

P r o o f. The proof follows from term-by-term application of convolution with the Abel kernel to relation (8) in view of the equality

$$\int_0^{x_1} \cdots \int_0^{x_n} \left(x_1^{k_1} - t_1^{k_1}\right)^{\frac{\mu_1-1}{k_1}} \cdots \left(x_n^{k_n} - t_n^{k_n}\right)^{\frac{\mu_n-1}{k_n}} \times$$

$$\times f_{p,k}\left(\lambda t_1, \lambda t_2 \ldots, \lambda t_n\right) dt_1 \ldots dt_n =$$

$$= \frac{1}{\lambda^{\mu_1} \cdots \lambda^{\mu_n}} \int_0^{\lambda x_1} \cdots \int_0^{\lambda x_n} \left((\lambda x_1)^{k_1} - \tau_1^{k_1}\right)^{\frac{\mu_1-1}{k_1}} \cdots \times$$

$$\times \left((\lambda x_n)^{k_n} - \tau_n^{k_n}\right)^{\frac{\mu_n-1}{k_n}} f_{p,k}\left(\tau_1, \tau_2, \ldots, \tau_n\right) d\tau_1 \ldots d\tau_n.$$

Theorem 7 is proved.

Euler's criterion

Let us recall Euler's theorem on homogeneous functions [10]:

Euler's theorem (Euler's criterion for homogeneous functions). *If the function* $f(x_1, x_2, \ldots, x_n)$ *is continuously differentiable in any point of space* R^n, *then for it to be Euler-homogeneous of degree p, it is necessary and sufficient that in any point of space* R^n *the following condition is satisfied*

$$x_1 \partial f/\partial x_1 + x_2 \partial f/\partial x_2 + \ldots +$$
$$+ x_n \partial f/\partial x_n = pf. \tag{16}$$

Relation (13) is obtained through differentiation of identical equation

$$f(\lambda x_1, \lambda x_2, \ldots, \lambda x_n) = \lambda^p f(x_1, x_2, \ldots, x_n)$$

for a homogeneous function of degree p with respect to parameter λ in point $\lambda = 1$, therefore, its necessity is obvious. However, it is highly nontrivial that condition (16) is not only necessary but sufficient for the function $f(x_1, x_2, \ldots, x_n)$ differentiable everywhere to be Euler-homogeneous of degree p. The proof of this theorem can be found, for example, in monograph [10].

Euler's criterion (16) works for continuously differentiable positively homogeneous functions of degree p as well. The only difference is that in this case the function $f(x_1, x_2, \ldots, x_n)$ can have

no derivative at point $x_1 = x_2 = \ldots = x_n = 0$ and, consequently, condition (16) is violated at this point.

Theorem 8 (generalization of Euler's criterion). *For the functions* $f_{p,k}(x_1, x_2, \ldots, x_n)$ *continuously differentiable everywhere to form a chain of fundamental associated homogeneous functions of degree p and order k, it is necessary and sufficient that the following equations are fulfilled at all points of space* R^n, *possibly except for point* $x_1 = x_2 = \ldots = x_n = 0$:

$$x_1 \partial f_{p,k}/\partial x_1 + x_2 \partial f_{p,k}/\partial x_2 + \ldots +$$
$$+ x_n \partial f_{p,k}/\partial x_n = p f_{p,k} + q f_{p,k-1}. \tag{17}$$

P r o o f. The necessity of relation (17) follows from differentiation of relation (9) as a composite function of λ at point $\lambda = 1$ (continuous differentiability is required here so that we could safely differentiate relation (9) as a composite function). The remaining task is to prove the sufficiency of relation (17).

When $k = 0$, the sufficiency of criterion (17) follows from Euler's theorem on homogeneous functions. Next, we apply the method of mathematical induction.

Suppose the statement is proved for all values of the index k in the interval $0 \le k \le m - 1$. Consider the function

$$\Phi_m(\lambda) = f_{p,m}(\lambda x_1, \lambda x_2, \ldots, \lambda x_n)/\lambda^p -$$
$$- \Sigma_{k=0,m} f_{p,m-k}(x_1, x_2, \ldots, x_n) (q \ln \lambda)^k/k!,$$

with summation carried out with respect to the index $1 \le k \le m$.

This expression coincides with identity (9), whose right and left-hand sides were divided by λ^p, up to the substitution of the summation index. The derivative of the function $\Phi_m(\lambda)$ with respect to the parameter λ is transformed to

$$d\Phi_m(\lambda)/d\lambda = (1/\lambda^{p+1}) [\lambda x_1 \partial f_{p,m}(\lambda x)/\partial x_1 +$$
$$+ \lambda x_2 \partial f_{p,m}(\lambda x)/\partial x_2 + \ldots +$$
$$+ \lambda x_n \partial f_{p,m}(\lambda x)/\partial x_n -$$
$$- p f_{p,m}(\lambda x) - q f_{p,m-1}(\lambda x) +$$
$$+ q f_{p,m-1}(\lambda x) - q \Sigma_{k=1,m} f_{p,m-k}(x) \lambda^p \times$$
$$\times (q \ln \lambda)^{k-1}/(k-1)!] = 0,$$

because relation (17) for the function $f_{p,m}(x_1, x_2, \ldots, x_n)$ is fulfilled, including at point $(\lambda x_1, \lambda x_2, \ldots, \lambda x_n)$, and the function

$f_{p,m-1}(x_1, x_2, ..., x_n)$ satisfies condition (9) by inductive assumption. Therefore, $\Phi_m(\lambda) = \mathrm{const}$ and, in particular, $\Phi_m(\lambda) = \Phi_m(1)$.

However, as it is easy to verify, the condition $\Phi_m(\lambda) = \Phi_m(1)$ means that relation (9) is fulfilled for the functions $f_{p,m}(x_1, x_2, ..., x_n)$. Consequently, if condition (17) is satisfied for $\forall k \geq 0$ at all points of space R^n, except possibly the origin of coordinates, then relation (9) is satisfied for $\forall k \geq 0$.

Theorem 8 is proved.

Note. To provide the condition $\Phi_m(\lambda) = \Phi_m(1) = \mathrm{const}$, the derivative $\Phi'_m(\lambda)$ must exist and become zero at any point of the segment connecting the points $(\lambda x_1, \lambda x_2, ..., \lambda x_n)$ and $(x_1, x_2, ..., x_n)$. If equality (14) is violated for at least one intermediate point, or at least the derivative $\Phi'_m(\lambda)$ exhibits discontinuities at one intermediate point, then the function $\Phi_m(\lambda)$ can be decomposed into piecewise constant steps. This is exactly why violation of continuous differentiability of the function at zero provides only positive Euler homogeneity for the function $f(x_1, x_2, ..., x_n)$, and not the general Euler homogeneity.

Theorem 9 (on integrating fundamental associated homogeneous functions). *If $f_{p,k}(x_1, x_2, ..., x_n)$ is a chain of fundamental associated homogeneous functions of degree p and order k, then there exist such functions $g_k(x_2, x_3, ..., x_n)$ for which the functions*

$$F_{p,k}(x_1, x_2, ..., x_n) =$$

$$= \int_{a_k}^{x_1} f_{p,k}(t, x_2, x_3, ..., x_n)\, dt + g_k(x_2, x_3, ..., x_n)$$

form a chain of fundamental associated homogeneous functions of degree $p + 1$ and order k.

Naturally, any coordinate x_i can be used instead of the coordinate x_1.

P r o o f. According to Theorem 6, it is necessary and sufficient that relations (17) are fulfilled for the functions $F_{p,k}(x_1, x_2, ..., x_n)$. This leads to the equation

$$0 = x_1 f_{p,k}(x_1, x_2, ..., x_n) +$$

$$+ \int_{a_k}^{x_1}\left(t\frac{\partial f_{p,k}}{\partial t} + x_2\frac{\partial f_{p,k}}{\partial x_2} + \cdots x_n\frac{\partial f_{p,k}}{\partial x_n} \right) dt -$$

$$- \int_{a_k}^{x_1} t\frac{\partial f_{p,k}}{\partial t}\, dt - (p+1)\int_{a_k}^{x_1} f_{p,k}(t, x_2, x_3, ...)\, dt -$$

$$- q\left(\int_{a_{k-1}}^{a_k} f_{p,k-1}(t, x_2, x_3, ...)\, dt +\right.$$

$$+ \left.\int_{a_k}^{x_1} f_{p,k-1}(t, x_2, x_3, ...)\, dt\right) + x_2\frac{\partial g_k}{\partial x_2} + x_3\frac{\partial g_k}{\partial x_3} +$$

$$+ \cdots x_n\frac{\partial g_k}{\partial x_n} - (p+1)g_k(x_2, x_3, ..., x_n) -$$

$$- qg_{k-1}(x_2, x_3, ..., x_n) =$$

$$= x_1 f_{p,k}(x_1, x_2, ..., x_n) - \int_{a_k}^{x_1}\left(t\frac{\partial f_{p,k}}{\partial t} + f_{p,k} \right) dt -$$

$$- q\int_{a_{k-1}}^{a_k} f_{p,k-1}(t, x_2, x_3, ..., x_n)\, dt +$$

$$+ x_2\frac{\partial g_k}{\partial x_2} + \cdots x_n\frac{\partial g_k}{\partial x_n} -$$

$$- (p+1)g_k(x_2, ..., x_n) -$$

$$- qg_{k-1}(x_2, ..., x_n) =$$

$$= x_2\frac{\partial g_k}{\partial x_2} + x_3\frac{\partial g_k}{\partial x_3} + \cdots x_n\frac{\partial g_k}{\partial x_n} -$$

$$- (p+1)g_k - qg_{k-1} -$$

$$- q\int_{a_{k-1}}^{a_k} f_{p,k-1}(t, x_2, x_3, ..., x_n)\, dt +$$

$$+ a_k f_{p,k}(a_k, x_2, x_3, ..., x_n).$$

The variable x_1 is absent in the obtained equations. Moreover, the function $g_{k-1}(x_2, x_3, ..., x_n)$ is already known. The remaining task is to find the solution to the equation

$$x_2\,\partial g_k/\partial x_2 + x_3\,\partial g_k/\partial x_3 + ... +$$

$$+ x_n\,\partial g_k/\partial x_n - (p+1)g_k \qquad (18)$$

$$= G_k(x_2, x_3, ..., x_n),$$

where the function $G_k(x_2, x_3, ..., x_n)$ is already known at the kth step of integration:

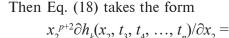$$G_k(x_2, x_3, ..., x_n) = qg_{k-1}(x_2, x_3, ..., x_n) +$$

$$+ q\int_{a_{k-1}}^{a_k} f_{p,k-1}(t, x_2, x_3, ..., x_n)\, dt -$$

$$- a_k f_{p,k}(a_k, x_2, x_3, ..., x_n).$$

It is convenient to use the following substitution to find this solution

$$g_k(x_2, x_3, ..., x_n) =$$

$$= x_2^{p+1} h_k(x_2, x_3/x_2, x_4/x_2, ..., x_n/x_2).$$

Then Eq. (18) takes the form

$$x_2^{p+2}\partial h_k(x_2, t_3, t_4, \ldots, t_n)/\partial x_2 =$$

$$= G_k(x_2, t_3 x_2, t_4 x_2, \ldots, t_n x_2).$$

A particular solution of this equation is found by transferring the multiplier x_2^{p+2} into the right-hand side and integrating the result with respect to the variable x_2 with 'frozen' variables t_3, t_4, ..., t_n. Furthermore, we need to add the general solution of homogeneous Eq. (18) with a zero right-hand side to the obtained particular solution of the inhomogeneous equation, that is, an Euler-homogeneous function of degree $(p + 1)$ depending on the variables x_2, x_3, ..., x_n.

Theorem 9 is proved.

As a result, we managed to not only prove that the required function $g_k(x_2, x_3, \ldots, x_n)$ exists but also to define its explicit quadratic form. The final solution is the sum of a particular case of the chain of functions $g_k(x_2, x_3, \ldots, x_n)$ expressed recursively in quadratic form in terms of the functions $f_{p,k}(a_k, x_2, x_3, \ldots, x_n)$, and a random chain of fundamental associated homogeneous functions of degree $(p + 1)$ and order k of the variables x_2, x_3, ..., x_n, which can be given explicitly using Eqs. (10), (11), (14) or (15).

Problem. Suppose for all points of space R^n, except possibly the point $x_1 = x_2 = \ldots = x_n = 0$, that the continuous differentiable functions $g_k(x_1, x_2, \ldots, x_n)$ satisfy the equalities

$$x_1 \partial g_k/\partial x_1 + x_2 \partial g_k/\partial x_2 + \ldots + \tag{19}$$
$$+ x_n \partial g_k/\partial x_n = p_k g_k + q_k g_{k-1},$$

where p_k, q_k are the given constants, and the functions $g_k(x_1, x_2, \ldots, x_n)$ with negative subscripts are taken to equal zero. What can we say about the form of the functions $g_k(x_1, x_2, \ldots, x_n)$?

If $\forall k$, $p_k = p = $ const, and $q_k = q = $ const, Euler's criterion (17) provides an answer immediately: the functions $g_k(x_1, x_2, \ldots, x_n)$ are fundamental associated homogeneous functions $f_{p,k}(x_1, x_2, \ldots, x_n)$ of degree p and order k. In the general case, additional calculations are required. After substituting

$$g_k(x_1, x_2, \ldots, x_n) =$$

$$= h_k(\ln x_1, x_2/x_1, x_3/x_1, \ldots, x_n/x_1),$$

the chain of conditions (19) is reduced to a system of ordinary linear differential equations with constant factors and a bidiagonal matrix of factors, where $t = \ln x_1$ is a free variable, and

the variables $t_2 = x_2/x_1$, $t_3 = x_3/x_1$, ..., $t_n = x_n/x_1$ are 'frozen'.

After solving the obtained system of differential equations and making the reverse transition to the variables x_1, x_2, ..., x_n, we obtain the general form of the functions $g_k(x_1, x_2, \ldots, x_n)$. At the same time, it should be borne in mind that the free constants obtained after integrating a system of ordinary linear differential equations with constant factors are in fact random functions depending on temporary 'frozen' variables $t_2 = x_2/x_1$, $t_3 = x_3/x_1$, ..., $t_n = x_n/x_1$. Depending on what the constants p_k are equal to and how many of them turn out to be equal to each other, the structure of the solution can be quite complicated.

In a particular case, let us take a chain of relations (19), where all values of p_k equal the same number p, while $\forall q_k \neq 0$. Then, according to condition (17), the functions $g_k(x_1, x_2, \ldots, x_n)$, scaled up by c_k times, turn out to be fundamental associated homogeneous functions $f_{p,k}(x_1, x_2, \ldots, x_n)$ described by the general equations (10) and (11) (or (14), or (15)), if the relations $c_k q_k / c_{k-1} = q$ are fulfilled (where the value of the parameter $q \neq 0$ is chosen arbitrarily). In other words, scaling factors c_k should be chosen in accordance with the recursive rule $c_k = q c_{k-1}/q_k$, where $c_0 = 1$, and the results coincide with a certain chain of fundamental associated homogeneous functions $f_{p,k}(x_1, x_2, \ldots, x_n)$ of degree p and order k up to the multipliers.

Differentiation with respect to degree of homogeneity

An interesting technique allowing to generate new fundamental associated homogeneous functions is considered in [11, 12]. Specifically, suppose $f_p(x_1, x_2, \ldots, x_n)$ is a one-parameter family of Euler-homogeneous functions with the degree of homogeneity equal to p, where p is a continuously changing parameter.

Repeatedly differentiating the homogeneity relation

$$f_p(\lambda x_1, \lambda x_2, \ldots, \lambda x_n) = \lambda^p f_p(x_1, x_2, \ldots, x_n)$$

with respect to the parameter p, we obtain that the functions

$$f_{p,k}(x_1, x_2, \ldots, x_n) = (1/k!) \, \partial^k f_p(x_1, x_2, \ldots, x_n)/\partial p^k$$

satisfy functional relations (9), i.e., are a particular case of fundamental associated homogeneous functions.

The homogeneous function $f_p(x_1, x_2, \ldots, x_n)$ can be represented using Eqs. (3) and (4):

$$\text{if } x_1 > 0, f_p(x_1, x_2, \ldots, x_n) =$$

$$= x_1^p \, h_p(x_2/x_1, x_3/x_1, \ldots, x_n/x_1); \tag{20}$$

$$\text{if } x_1 < 0, f_p(x_1, x_2, \ldots, x_n) =$$

$$= (-x_1)^p \, g_p(x_2/x_1, x_3/x_1, \ldots, x_n/x_1), \tag{21}$$

where $h_p(t_2, t_3, \ldots, t_n)$, $g_p(t_2, t_3, \ldots, t_n)$ are functions of $(n-1)$ variables independent of each other.

These functions are mapped one-to-one with respect to the given function $f_p(x_1, x_2, \ldots, x_n)$ according to the formulae

$$h_p(t_2, t_3, \ldots, t_n) = f_p(+1, t_2, t_3, \ldots, t_n),$$

$$g_p(t_2, t_3, \ldots, t_n) = f_p(-1, -t_2, -t_3, \ldots, -t_n),$$

and depend on the continuous parameter p as well.

Repeatedly differentiating expressions (20), (21) with respect to the parameter p, we obtain the universal formulae (10), (11) for fundamental associated homogeneous functions, if the new functions $h_j(x_1, x_2, \ldots, x_n)$ and $g_j(x_1, x_2, \ldots, x_n)$ are defined as

$$h_j(t_2, t_3, \ldots, t_n) = (1/j!) \, \partial^j h_p(t_2, t_3, \ldots, t_n)/\partial p^j,$$

$$g_j(t_2, t_3, \ldots, t_n) = (1/j!) \, \partial^j g_p(t_2, t_3, \ldots, t_n)/\partial p^j.$$

Eqs. (15) are obtained similarly by differentiating the function $f_p(x_1, x_2, \ldots, x_n)$ with respect to the parameter p. The function is written as

$$f_p(x_1, x_2, \ldots, x_n) = r^p \, h_p(x_1/r, x_2/r, \ldots, x_n/r),$$

where $r = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$, and $h_p(t_1, t_2, \ldots, t_n)$ is a real function given on the surface of a unit hypersphere

$$t_1^2 + t_2^2 + \ldots + t_n^2 = 1$$

and related to the function $f_p(x_1, x_2, \ldots, x_n)$ by

$$h_p(t_1, t_2, \ldots, t_n) = f_p(t_1, t_2, \ldots, t_n),$$

where $t_1^2 + t_2^2 + \ldots + t_n^2 = 1$.

It follows from the obtained formulae that the process of differentiating the Euler-homogeneous functions with the degree of homogeneity equal to p with respect to the continuously changing parameter p does not generally lead to a loss of possible chains of fundamental associated homogeneous functions.

Importantly, if the functions $f_p(x_1, x_2, \ldots, x_n)$ are harmonic (or fulfil some other linear differential equation in partial derivatives with constant coefficients), then all the fundamental associated homogeneous functions obtained by differentiating the initial function $f_p(x_1, x_2, \ldots, x_n)$ with respect to the parameter p are also harmonic.

Conclusion

Analyzing mutually homogeneous functions which correspond to a matrix of functional equations with identical real eigenvalues, we obtained a refined class of associated homogeneous Gel'fand functions [11, 12]. The definitions and theorems formulated in the study allow to correctly describe this important class of functions and consider its properties in detail. In particular, Theorem 2 on fundamental associated homogeneous functions allows to safely consider the following generalizations

$$f_{p,k}(x_1, x_2, \ldots, x_n) =$$

$$= (1/k!) \, R_p(x_1, x_2, \ldots, x_n) \times$$

$$\times (\ln S_q(x_1, x_2, \ldots, x_n))^k$$

and argue that such functions identically coincide with the given class of functions, while fully preserving their properties without producing any fundamentally new mathematical objects.

The mathematical constructions we have discussed may prove useful not only for theoretical studies but also for practical applications. The property of Euler homogeneity for scalar potentials of electric and magnetic fields [5−8] allows to synthesize efficient electron and ion-optical systems, presented, for example, in a series of works by Khursheed [27−43].

We hope that the obtained functional constructions generalizing the relation of Euler homogeneity can make it possible to transfer the principle of trajectory similarity, introduced by Golikov [5−8], to wider classes of electric and magnetic fields.

The calculations in this paper were carried out using the Wolfram Mathematica software [44].

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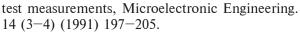1. **Berdnikov A.S., Gall L.N., Gall N.R., Solovyev K.V.**, Generalization of the Thomson formula for harmonic functions of a general type, St. Petersburg Polytechnical State University Journal. Physics and Mathematics. 12 (2) (2019) 32−48.

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