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FRACTIONAL DIFFERENTIATION OPERATION IN THE FOURIER BOUNDARY PROBLEMS

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We use the algebra of unbounded differentiation t operators acting on the ring of differentiable functions. The analytical representation of the fractional degree of the t operator is used to construct the resolvents of three boundary problems for the Fourier equation. Periodic solutions of limiting Fourier problems in the algebra of differentiation operators coincide with classical solutions. The $t + 2$ extension is a continuous spectrum of the Fourier transform and allows us to obtain exact solutions of three limit problems for a domain of any dimension $d>1$.

Keywords: differential equation, Abel–Liouville formula, ring of operators, inverse operator, carrier, distribution

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ОПЕРАЦИЯ ДРОБНОГО ДИФФЕРЕНЦИРОВАНИЯ В ПРЕДЕЛЬНЫХ ЗАДАЧАХ ФУРЬЕ

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Применяется алгебра неограниченных операторов дифференцирования t, действующая над кольцом дифференцируемых функций. Аналитическое представление дробной степени оператора t используется для построения резольвент трех предельных задач для уравнения Фурье. Периодические решения предельных задач Фурье в алгебре операторов дифференцирования совпадают с классическими решениями. Расширение $t+2$ – непрерывный спектр преобразования Фурье, позволяет получить точные решения трех предельных задач для области любой размерности $d > 1$.

Ключевые слова: дифференциальное уравнение, формула Лиувилля, кольцо операторов, обратный оператор, носитель, распределение

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Introduction

The classic theory of thermal stability of wall structures developed by Seliverstov [1] uses the methods of Fourier series theory and, in a certain sense, originates from these methods. This is hardly accidental, as the author of the study was an expert in Fourier series theory. The methods of trigonometric series are sufficient if the boundary temperature distributions of external sources belong to L_p ($p > 1$) on a set of times t. Fourier series converge almost everywhere on such a set. However, the above condition is redundant for applied problems. While the temperature distribution of the sources is typically continuous at best, according to Titchmarsh [2], it was impossible to prove similar statements for convergence of Fourier series almost everywhere [2, pp. 420–421]. The methods for expanding Fourier series are inconvenient for mixed boundary problems, especially if the external heat source depends on parameter t (time).

This study focuses on the methods for solving boundary problems for the Fourier equation in the form of equalities containing functions of differential operators, comparing the distributions obtained with the known exact solutions.

The significance of our study is in offering potential solutions for solving problems related to thermal stability of construction barriers.

Key approaches to obtaining the solutions

We have formulated and proved the following statements.

1. The solutions for the second and third boundary problems for the Fourier equation are obtained by solving the first boundary problem inverting the differentiation operator.

2. Support measures for the distribution of the primitive $x(t,s)$, δ_x and the primitive derivative $y:=-\partial x/\partial s$, δ_y satisfy the inequality $\delta_y/\delta_x \geq 1$ in the first-kind boundary problem.

3. Increasing the dimension of the domain does not increase the support measures of the distribution.

Statement 3 implies that the thermal resistance of a half-space does not exceed the thermal resistance of a half-plane. In turn, the thermal resistance of a half-plane does not exceed the thermal resistance of a half-line.

As an auxiliary technique, we use the following representation of the Taylor series (shift) for functions $f(t)$, analytic on half-line $t > 0$, $f \in C^\infty(0,\infty)$:

$$f(t+s)=\exp(s\partial_t)f(t),$$

and its inversion

$$f(t)=\exp(-s\partial_t)f(t+s),$$

containing integer powers of the differential operator ∂_t.

Simple expressions for measures of the supports $\delta_{x,y}$

Using the operator norms of fractional powers of the operator ∂_t allows to obtain simple expressions for measures of the supports $\delta_{x,y}$.

Preliminary considerations. Fractional differentiation is related to the solution of the Cauchy problem for an ordinary differential equation of arbitrary positive integer (natural) order $s > 0$.

Let

$$t \in \mathfrak{D}(x) \subset \mathcal{C}^{(s)}\left(R^1\right), y \in \mathcal{J}(x) \subseteq \mathcal{L}_1^{(loc)}\left(R^1\right).$$

then the Cauchy problem

$$\begin{gathered}\partial_t^s x = y, \partial_t^r x(0) = 0,\\ r = 0(1)s-1, \partial_t := d/dt\end{gathered} \tag{1}$$

has the following solution [3]:

$$x(t)=\frac{1}{(s-1)!}\int_0^t (t-\tau)^{s-1} y(\tau)d\tau, \tag{2}$$

or, in symbolic form,

$$x(t)=\partial_t^{-s} y(t). \tag{2a}$$

Given non-integer s, Eq. (2) can be extended:

$$\begin{gathered}\partial_t^{-s} y(t)=\frac{1}{\Gamma(s)}\int_0^t (t-\tau)^{s-1} y(\tau)d\tau,\\ \Gamma(s):=(s-1)!, s>0.\end{gathered} \tag{2b}$$

If $s = \sigma + i\rho$, $\sigma > 0$, Eq. (2b) takes the form

$$\begin{gathered}\partial_t^{-(\sigma+i\rho)} y(t)=\frac{1}{\Gamma(\sigma+i\rho)}\times\\ \times\int_0^t (t-\tau)^{\sigma-1}\left(\cos\left(\rho\ln(t-\tau)\right)+\right.\\ \left.+i\sin\left(\rho\ln(t-\tau)\right)\right) y(\tau)d\tau.\end{gathered}$$

Let $s=1/2$. Then, by virtue of expression (2b), we obtain Abel's formula:

$$\partial_t^{-1/2} y(t) = \frac{1}{\sqrt{\pi}} \int_0^t \frac{y(\tau)}{\sqrt{t-\tau}} d\tau = = \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{t}} y(t-z^2) dz. \tag{2c}$$

The formula obtained (2c) can be used to calculate the derivatives of the powers of t, for instance,

$$\partial_t^{-1/2} 1 = 2\sqrt{t/\pi}, \partial_t^{1/2} 1 = 1/\sqrt{t\pi},$$

$$\partial_t^{-1/2} t = \frac{4\sqrt{t^3}}{3\sqrt{\pi}}, \partial_t^{1/2} t = \frac{2\sqrt{t}}{\sqrt{\pi}},$$

furthermore, for any $n > 0$:

$$\partial_t^{-1/2} t^n = t^{n+1/2} \frac{\Gamma(n+1)}{\Gamma(n+3/2)},$$

$$\partial_t^{1/2} t^n = t^{n-1/2} \frac{(n+1/2)\Gamma(n+1)}{\Gamma(n+3/2)}.$$

Clearly, the kernel of the operator ∂_t^{-s}, $N(\partial_t^{-s})$, contains only one element, y = 0, for any $0 < s < 1$.

Commutation. By definition, the following expression holds true:

$$\partial_t^{1/2} y(t) = \frac{1}{\sqrt{\pi}} \frac{d}{dt} \int_0^t \frac{y(t-\tau)}{\sqrt{\tau}} d\tau = = \frac{y(0)}{\sqrt{t\pi}} + \frac{1}{\sqrt{\pi}} \int_0^t \frac{\partial_t y(t-\tau)}{\sqrt{\tau}} d\tau = = \frac{y(0)}{\sqrt{t\pi}} + \partial_t^{-1/2}(\partial_t y(t)),$$

or

$$\left(\partial_t \partial_t^{-1/2} - \partial_t^{-1/2} \partial_t\right) y(t) = \frac{y(0)}{\sqrt{t\pi}}. \tag{3}$$

If $y(0) = 0$, then the operator ∂_t commutes with its negative fractional power, e.g., −1/2:

$$\left(\partial_t \partial_t^{-1/2} - \partial_t^{-1/2} \partial_t\right) y(t) = 0, \tag{3a}$$

or, in symmetric form,

$$\partial_t^{-1/2} = \partial_t^{-1} \partial_t^{-1/2} \partial_t, \partial_t = \partial_t^{-1/2} \partial_t \partial_t^{1/2}.$$

It follows then that the operator ∂_t and its fractional powers are self-similar in case of commutation.

If

$$y(t \pm t_0) - y(t) = 0, \forall |t| > 0, t_0 > 0$$

in the Cauchy problem (1) is a primitive period, and a periodic solution is sought, so that

$$x(t \pm t_0) - x(t) = 0, \forall |t| > 0,$$

the periodic condition can be replaced by the homogeneous condition [3]:

$$\partial_t^r x(-\infty) = 0, \; r = 0(1)s-1,$$

and then the solution to the periodic Cauchy problem takes the form

$$\partial_t^{-s} x(t) = \frac{1}{\Gamma(s)} \int_{-\infty}^t (t-\tau)^{s-1} y(\tau) d\tau = = \frac{1}{\Gamma(s)} \int_0^\infty \omega^{s-1} y(t-\omega) d\omega.$$

Let $s = 1/2$, and then the previous formula takes the form

$$\partial_t^{-1/2} x = \frac{2}{\sqrt{\pi}} \int_0^\infty y(t-z^2) dz. \tag{2d}$$

Thus, the commutator in the periodic boundary problem equals zero, and the fractional power of the operator ∂_t is permutable with its inverse power.

Relations (2)–(2d) are known as the Abel–Liouville identities [13]. Applications to different mechanics problems are presented in Caputo's study (unfortunately, the original text was unavailable to us but it is cited in many later studies, for example, in [5–17] and references therein).

Extension 1. For any $s > 0$, inversion of the fractional differential operator has the form

$$\partial_t^{-s} x = \frac{1}{\Gamma(s+1)} \int_0^{t^s} y(t - z^{1/s}) dz$$

for a non-periodic problem and

$$\partial_t^{-s} x = \frac{1}{\Gamma(s+1)} \int_0^\infty y(t - z^{1/s}) dz$$

for a periodic one.

Indeed, if

$$y(t \pm 1) - y(t) = 0, \forall |t| > 0,$$

then the Cauchy condition for all derivatives takes the Lyapunov form:

$$\partial_t^s x(-\infty) = 0.$$

Extension 2. Let us consider an equation depending on the parameter λ:

$$(\partial_t - \lambda) x(t) = y(t).$$

Evidently, the kernel $N(\partial_t - \lambda)$ of the operator $\partial_t - \lambda$ consists of the exponents $x(t) = \exp(\lambda t)$. Therefore, the solution to the equation is

$$x = (\partial_t - \lambda)^{-1} y + z, z \in \mathfrak{N}(\partial_t - \lambda).$$

The equation

$$(\partial_t - \lambda)^n x(t) = y(t)$$

has the solution

$$x(t) = (\partial_t - \lambda)^{-n} y(t) + (z),$$

$$(z) \in \mathfrak{N}((\partial_t - \lambda)^n);$$

evidently,

$$\mathfrak{N}(\partial_t - \lambda) \subset \mathfrak{N}(\partial_t - \lambda)^2 \subset \ldots \subset \mathfrak{N}(\partial_t - \lambda)^n.$$

Integral representation of the solution to the homogeneous Cauchy problem has the form

$$x(t) = \frac{1}{(n-1)!} \int_0^t (t-\tau)^{n-1} \exp(\lambda(t-\tau)) y(\tau) d\tau.$$

Here the kernel consists of functions

$$P_{n-1}(t) \exp(\lambda t) = z(t) \subset \mathfrak{N}(\partial_t - \lambda)^n,$$

where $P_s(t)$ is a polynomial of degree s.

Let us continue solving the homogenous Cauchy problem for fractional values of n:

$$x(t) = (\partial_t - \lambda)^{-n} y(t) =$$

$$= \frac{1}{\Gamma(n)} \int_0^t (t-\tau)^{n-1} \exp(\lambda(t-\tau)) y(\tau) d\tau.$$

Let $n=1/2$; then

$$x(t) = (\partial_t - \lambda)^{-1/2} y(t) =$$

$$= \frac{1}{\sqrt{\pi}} \int_0^\tau \frac{\exp(\lambda(t-\tau))}{\sqrt{t-\tau}} y(\tau) d\tau =$$

$$= \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{t}} \exp(\lambda z^2) y(t - z^2) dz.$$

It is sufficient for integrals to converge that the following condition hold true for the real part of the number λ: Reλ < 0.

Similarly, the periodic solution takes the form

$$x(t) = (\partial_t - \lambda)^{-1/2} y(t) =$$

$$= \frac{2}{\sqrt{\pi}} \int_0^\infty \exp(\lambda z^2) y(t - z^2) dz.$$

Extension 3. Given arbitrary $n > 0$, the inversion equations for the fractional powers of the operator take the form:

$$x(t) = \frac{1}{\Gamma(n+1)} \int_0^{t^n} \exp(\lambda z^{1/n}) y(t - z^{1/n}) dz,$$

for a non-periodic problem and

$$x(t) = \frac{1}{\Gamma(n+1)} \int_0^\infty \exp(\lambda z^{1/n}) y(t - z^{1/n}) dz,$$

for a periodic problem, while Reλ < 0.

This extension is thoroughly explored in monograph [13] but the authors but apparently did not use the trivial substitution $\sqrt{t} = z$. This substitution is convenient because it allows to represent the fractional differential operator as a probability integral. Indeed, the integrand in Eq. (2d) can be expanded in a Taylor series:

$$x(t - z^2) = \exp(-z^2 \partial_t) x(t),$$

then the left-hand side of Eq. (2d) is obtained immediately.

Nash's and Kuiper's studies (discussed in Gromov's monograph [18]) formulated the so-called h-principle: differential operators R connecting partial derivatives are regarded as algebraic relations for partial derivatives.

The h-principle is substantiated in [18], accompanied by a list of publications up to 1990. Sobolev spaces of functions with (generalized) derivatives of fractional order were considered by Slobodetskii in a series of studies [19, 20], developing Bakelman's earlier ideas [21] on the geometric theory of equations.

Analysis of Fourier boundary problems for half-line $s > 0$

First boundary problem. Let us consider the first boundary problem in an unbounded domain $t > 0$, $s > 0$:

$$\frac{\partial x}{\partial t} = \frac{\partial^2 x}{\partial s^2}, x(t,0) = x_0(t). \quad (4)$$

We find a formal solution to this boundary problem by separation of variables.

Let

$$x(t,s)=\exp(-s\alpha)x_0(t), \quad (5)$$

While the parameter $\alpha > 0$, which guarantees a decrease in $x(t,s)$, uniform with respect to t. In this case, substituting equality (5) in the equation of problem (4) leads to the condition

$$\exp(-s\alpha)(\partial_t-\alpha^2)x_0(t)=0,$$

from which it follows that $\alpha = \partial^{1/2}_t$, and, by virtue of equality (5), the solution of boundary problem (4) has the form

$$x(t,s)=\exp\left(-s\partial_t^{1/2}\right)x_0(t). \quad (6)$$

***Verification of solution* (6).** *Step* 1. The classical solution for boundary problem (4) has the following form:

$$x(t,s)=\frac{2}{\sqrt{\pi}}\int_{\frac{s}{2\sqrt{t}}}^{\infty}x_0\left(t-\frac{s^2}{4z^2}\right)\exp\left(-z^2\right)dz. \quad (7)$$

Let us expand the function $x_0\left(t-\frac{s^2}{4z^2}\right)$ into a Taylor series in powers of $\frac{s^2}{4z^2}$:

$$x_0\left(t-\frac{s^2}{4z^2}\right)=\exp\left(-\frac{s^2\partial_t}{4z^2}\right)x_0(t).$$

Solution (7) then takes the form [4]:

$$x(t,s)=\frac{2}{\sqrt{\pi}}\int_{\frac{s}{2\sqrt{t}}}^{\infty}\exp\left(-z^2-\frac{s^2\partial_t}{4z^2}\right)dz\left(x_0(t)\right). \quad (7a)$$

However, it is known from the course on analysis of infinitely small quantities, developed by de la Vallée Poussin [3], that

$$\frac{2}{\sqrt{\pi}}\int_0^{\infty}\exp\left(-u^2-\alpha/u^2\right)du=\exp\left(-2\sqrt{\alpha}\right).$$

Therefore, if the lower limit in the integral in (7a) equals zero, then Eq. (7a) coincides with Eq. (6). Thus, Eq. (7a) takes the following form:

$$x(t,s)=\exp\left(-s\partial_t^{1/2}\right)x_0(t)-$$

$$-\frac{2}{\sqrt{\pi}}\int_0^{\frac{s}{2\sqrt{t}}}\exp\left(-z^2-\frac{s^2\partial_t}{4z^2}\right)dz\cdot x_0(t). \quad (7b)$$

Consequently, given that $\frac{s}{2\sqrt{t}}<<1$, Eqs. (7b) and (6) yield close results.

Step 2. If $x_0(t)$ is a periodic time function, i.e.,

$$x_0(t\pm t_0) = x_0(t),$$

where $t_0 < 0$ is a primitive period, then, instead of solutions (7), (7a) and (7b), we obtain a solution in the form

$$x(t,s)=\frac{2}{\sqrt{\pi}}\int_0^{\infty}x_0\left(t-\frac{s^2}{4z^2}\right)\exp\left(-z^2\right)dz, \quad (7c)$$

and solutions (7c) and (6) are then identical. To confirm this, it is sufficient to expand the integrand in solution (7c) in a Taylor series:

$$x(t,s)=\frac{2}{\sqrt{\pi}}\int_0^{\infty}\exp\left(-z^2-\frac{s^2\partial_t}{4z^2}\right)dz\cdot x_0(t)=$$

$$=\exp\left(-s\partial_t^{1/2}\right)x_0(t),$$

which proves the identity.

Thus, Eq. (6) and its corollaries hold true for a boundary value $x(t,0) = x_0(t)$, periodic with respect to the parameter t, i.e., for the solution of the quasi-steady boundary problem of thermal conductivity.

The second boundary problem. Eq. (6) implies that the derivative $y(t,s)=-\frac{\partial x}{\partial s}$ is calculated as follows:

$$y(t,s)=\partial_t^{1/2}\exp\left(-s\partial_t^{1/2}\right)x_0(t). \quad (8)$$

Let s=0. By virtue of expression (8),

$$y(t,0):=y_0(t)=\partial_t^{1/2}x_0(t),$$

$$x_0(t)=\partial_t^{-1/2}y_0(t),$$

and by virtue of solution (6), the solution to the second boundary problem takes the form

$$x(t,s)=\exp\left(-s\partial_t^{1/2}\right)\partial_t^{-1/2}y_0(t). \quad (9)$$

The third boundary problem. The given problem is formulated as follows for the Fourier equation:

$$\left(\frac{\partial x}{\partial s}\right)_{s=0}+\beta\left(x_e-x_0\right)=0, \quad (10)$$

where x_e is the potential of an external source, β is the transfer coefficient.

Equality (10) then takes the form

$$\left(\partial_t+\beta\right)x_0(t)=\beta x_e,$$

which implies that

$$x_0(t) = (\partial_t + \beta)^{-1}(\beta x_e),$$
$$x(t,s) = \exp\left(-s\partial_t^{1/2}\right)\left((\partial_t + \beta)^{-1}(\beta x_e)\right). \quad (11)$$

So, if the boundary parameters y_0, x_e, β are periodic time functions, then solutions (9) and (11) coincide with the classical solutions.

Measure of distribution supports for half-line *s* > 0

We define the support supp($x(t,s)$) of the distribution $x(t,s)$ as a set of values of the coordinate s on which the distribution $x(t,s)$ is concentrated. If the distribution $x(t,s)$ has continuous density, we can define the support as the thickness of the x-layer with respect to the density limit, $x_0(t)$:

$$\delta_x(t) := \frac{1}{x_0}\int_0^\infty x(t,s)\,ds.$$

By virtue of solution (6), the thickness of the x-layer is expressed as

$$\delta_x(t) = \frac{\partial_t^{-1/2} x_0(t)}{x_0(t)}.$$

If the distribution $x_0(t)$ is periodic, the given thickness follows the expression

$$\delta_x(t) = \frac{2}{\sqrt{\pi}x_0(t)}\int_0^\infty x_0\left(t - z^2\right)dz.$$

Similarly, the thickness of y-layer is expressed as

$$\delta_y(t) := \frac{1}{y_0(t)}\int_0^\infty y(t,s)\,ds = \frac{x_0(t)}{\partial_t^{1/2}x_0(t)} =$$
$$= \frac{\sqrt{\pi}}{2}\frac{x_0(t)}{\int_0^\infty \dot{x}\left(t - z^2\right)dz},$$

where the dot denotes the derivative with respect to the entire argument $t - z^2$.

Lemma 1. *The ratio between the layer thicknesses (form parameter), expressed by the formula*

$$\delta_y / \delta_x = \frac{x_0^2}{\partial_t^{1/2}x_0 \cdot \partial_t^{-1/2}x_0} =$$
$$= \frac{\pi x_0^2(t)}{2}\left(\frac{d}{dt}\left(\int_0^\infty x_0\left(t - z^2\right)dz\right)^2\right)^{-1},$$

has a value of at least unity for any bounded distribution $x_0(t)$.

P r o o f. Indeed, the above expression can be written as

$$\delta_y / \delta_x = \frac{\pi}{2}\frac{\partial_t^{-1}x_0^2}{\left(\partial_t^{-1}x_0\right)^2} \ge \frac{\pi}{2} > 1.$$

Here we use the Cauchy inequality to estimate the integrals.

To illustrate that the lemma proved holds true, let us provide an example which allows to calculate the support lengths directly. The distribution $x(t,s)$ for a straight line (ray) $s>0$ takes the form

$$x(t,s) = \operatorname{erfc}\left(\frac{s}{2\sqrt{t}}\right),$$

where

$$x(t,0) : x_0(t) - 1 = x(0,s) = 0.$$

Then we obtain the following equations:

$$y(t,s) := -\frac{\partial x}{\partial s} = \frac{1}{\sqrt{t\pi}}\exp\left(-\frac{s^2}{4t}\right),$$
$$y(t,0) := y_0(t) = \frac{1}{\sqrt{t\pi}},$$

$$\delta_x = 2\sqrt{\frac{t}{\pi}},$$
$$\delta_y = \sqrt{t\pi},$$
$$\delta_y / \delta_x = \pi / 2.$$

The lemma is proved.

Lemma 2. *Let*

$$f(x) := \int_x^\infty \exp\left(-at^m\right)dt, f(0) = \int_0^\infty \exp\left(-at^m\right)dt,$$

where a, m are positive constants, and

$$-f'(x) := \varphi(x) = \exp\left(-ax^m\right), -f(0) = 1.$$

Then the ratio between the support lengths of function $f(x)$ and its derivative $f'(x) = \phi(x)$ (δ_ϕ and δ_f, respectively) has a value no less than unity:

$$\mathcal{K} := \delta_\phi / \delta_f = \frac{1}{m}\frac{\left(\Gamma(1/m)\right)^2}{\Gamma(2/m)} \ge 1.$$

P r o o f. Indeed, the following equations hold true:

$$\delta_f = \frac{\int_0^\infty t\exp\left(-at^m\right)dt}{\int_0^\infty \exp\left(-at^m\right)dt,},$$

$$\delta_\varphi = \int_0^\infty \exp\left(-at^m\right)dt,$$

$$\mathcal{K} = \frac{\left(\int_0^\infty \exp\left(-at^m\right)dt\right)^2}{\int_0^\infty t\exp\left(-at^m\right)dt},$$

and it remains to rewrite the integrals in Euler form.

The Lemma is proved.

The results of Lemma 2 can be rewritten differently, if we use a duplication formula for the function $\Gamma(z)$ [2, 3]:

$$\mathcal{K} = \frac{\pi}{2^{2/m-1}m}\frac{\Gamma(1/m)}{\Gamma(1/m+1/2)}.$$

Let $m = 1$, then $\mathcal{K} = 1$; if $m = 2$, then $\mathcal{K} = \pi/2$. It is easy to use the asymptotic form of the Γ-function to prove that $\mathcal{K} \underset{m\to\infty}{\to} \infty$.

Thus, the measure (length) of the distribution support for decreasing integer distributions of the order $m > 1$, the measure (length) of the distribution support does not exceed the measure of the distribution support derivative.

The quantity δ_y/δ_x in problems of thermal conductivity of wall structures is the ratio of absolute to effective thermal resistance of one-dimensional heat conducting medium (of the half-line $s > 0$)[22].

Fourier boundary problems for half-plane $s > 0$, $|u| < \infty$

Let

$$D(x) = (t,s,u\text{: } t>0, s>0, |u|<\infty),$$

where u is the second coordinate.

The Fourier equation

$$\frac{\partial x}{\partial t} = \nabla^2_{s,u}x$$

and the first-kind boundary condition

$$x(t,0,u) = x_0(t,u)$$

are satisfied.

We define the transformation

$$x(t,s,u),\ \hat{x} = \hat{x}(t,s)$$

that is integral with respect to the argument u as

$$\hat{x}(t,s) := \int_{-\infty}^{\infty} x(t,s,v)\exp(i\omega v)dv,$$

where the circumflex ^ denotes the Fourier transform of the function $x(t,s,u)$ with respect to the argument u.

The Fourier transform of the function $x(t,s,u)$ satisfies the partial differential equation:

$$\left(\frac{\partial}{\partial t} + \omega^2\right)\hat{x} = \frac{\partial^2 \hat{x}}{\partial s^2}. \qquad (12)$$

Eq. (11) can be obtained from Eq. (4) by replacing the operator ∂_t with the operator

$$\partial_{t,\omega} = \partial_t + \omega^2,$$

where ω is the spectral number.

The first-kind boundary condition is formulated as

$$\hat{x}(t,0) = \hat{x}_0(t). \qquad (13)$$

Then, similar to solution (6), we obtain:

$$\hat{x}(t,s) = \exp\left(-s\partial^{1/2}_{t,\omega}\right)\hat{x}_0(t). \qquad (6a)$$

Next, the solution to the second boundary problem has the form

$$\hat{x}(t,s) = \exp\left(-s\partial^{1/2}_{t,\omega}\right)\partial^{-1/2}_{t,\omega}\hat{y}_0(t),$$

$$\hat{y}_0(t) := -\left(\frac{\partial \hat{x}}{\partial s}\right)_{s=0}. \qquad (9a)$$

Finally, the solution to the third boundary problem follows the expression

$$\hat{x}(t,s) = \exp\left(-s\partial^{1/2}_{t,\omega}\right)\times$$

$$\times\left(\left(\partial_{t,\omega} + \hat{\beta}\right)^{-1}\left(\hat{\beta x_e}\right)\right). \qquad (11a)$$

As a result, Eqs. (6a), (9a) and (11a) coincide with the exact solutions of the periodic boundary problems and are obtained from the solutions to one-dimensional problems by replacing the operator ∂_t with the operator $\partial_{t,\omega}$.

Generalization of analysis. The Fourier equation with respect to the coordinates $s, u_1, \ldots, u_{d-1}$ for the case $d > 1$ has the following form:

$$\frac{\partial x}{\partial t} = \frac{\partial^2 x}{\partial s^2} + \sum_{i=1}^{d-1} \frac{\partial^2 x}{\partial u_i^2},$$

Applying $(d - 1)$-fold Fourier transform, the equation is written as

$$\left(\frac{\partial}{\partial t} + \Omega^2\right)\hat{x} = \frac{\partial^2 \hat{x}}{\partial s^2}, \Omega^2 := \sum_{i=1}^{d-1} \omega_i^2.$$

The solution to the first boundary problem in traditional notation has the form

$$\hat{x}(t,s) = \exp\left(-s\partial_{t,\Omega}^{1/2}\right)\hat{x}_0, \qquad (6b)$$

where we introduce the following notation for $(d - 1)$-fold Fourier transform

$$\hat{x}_0(t, \omega_1, \ldots \omega_{d-1}) = \frac{1}{(2\pi)^{d-1}} \times$$

$$\times \int_0^\infty dv_1 \ldots \int_0^\infty dv_{d-1} x_0(t, 0, v_1, \ldots v_{d-1}) \times$$

$$\times \exp\left(i \sum_{i=1}^{d-1} \omega_i v_i\right).$$

The inverse Fourier transform should be represented as

$$x_0(t, 0, u_1, \ldots u_{d-1}) = \int_0^\infty d\omega_1 \ldots \int_0^\infty d\omega_{d-1}\, \hat{x}_0 \times$$

$$\times (t, \omega_1, \ldots \omega_{d-1}) \exp\left(-i \sum_{i=1}^{d-1} \omega_i u_i\right).$$

If $x_0(t,0,u_1,\ldots, u_{d-1})$ is a periodic function of the argument t, then Eq. (6b) coincides with the exact solution to the first Fourier boundary problem. Eqs. (9a) and (11a) also hold true if the subscript ω is replaced with Ω.

Let us return to the thermal conductivity problem mentioned at the end of the section "Measure of distribution supports for half-line $s > 0$". It can be proved that if the dimension of an infinite domain occupied by scalar heat-conducting medium increases, its thermal resistance does not increase with an increase in the dimension of the domain $d > 1$.

Indeed, for any value $d > 1$,

$$\left\| \partial_t + \sum_{1 \le i \le d-1} \omega_i^2 \right\|^{-s} \le \left\| \partial_t^{-s} \right\| \le \left\| \partial_t \right\|^{-s}.$$

Conclusion

Using the algebra of unbounded differentiation operators and reviewing the results of the analysis carried out, we have drawn the following conclusions.

1. The unbounded operator of fractional differentiation over a ring of continuous functions can be inverted (known as the Abel–Liouville formula). The inverse operator is bounded on functions from the set $L_1(0,t)$, where $t \le \infty$. The solutions for the second and third Fourier boundary problems are obtained by inverting the differentiation operator of the first boundary problem.

2. The operator ∂_t in a quasi-steady (periodic) boundary problem commutes with any fractional inverse power. There are no operator powers in aperiodic commutation problems.

3. In case of decreasing integer distributions of order $m > 1$, the support measure (length) of the distribution $x(t,s)$ does not exceed the support measure corresponding to the derivative of the distribution $y(t,s)=\partial x/\partial s$. In other words, the thickness of the heat flux boundary layer (decreasing distribution of order $m > 1$) should be no less than the thickness of the temperature boundary layer.

4. Increasing the dimension of the domain $D(x)$ of the sought-for function $x(t,s)$ does not increase the measures of the supports supp(x) and supp(y), where $y = \|\nabla x\|$ ($\|\nabla x\|$ is the Euclidian norm of the scalar function $x(t,s)$). The support measure (length) of the distribution $x(t,s)$ does not exceed the support measure corresponding to the derivative of the distribution for any decreasing integer distributions of order $m > 1$. Therefore, the thermal resistance of the domain $D(x)$ does not increase along with increasing dimension: the heat flux vector **y** gains an additional component (additional degree of freedom).

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