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MUTUALLY HOMOGENEOUS FUNCTIONS WITH FINITE-SIZED MATRICES

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This work continues our studies in the properties of the homogeneous Euler' s functions that can be used in the synthesis of electric and magnetic fields for electron and ion-optical systems to carry out spectrographic recording mode. A generalization of a functional general equation for homogeneous functions has been considered. This equation corresponds to linear functional relations with a minimal-sized matrix. A general solution of the obtained functional equation was found assuming of differentiability of the functions in question. The resulting systems of functions were termed mutually homogeneous functions by analogy with the homogeneous Euler's functions and the associated homogeneous Gel'fand's functions.

Keywords: functional equation, associated homogeneous function, mutually homogeneous functions, spectrograph

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ВЗАИМНО-ОДНОРОДНЫЕ ФУНКЦИИ С МАТРИЦАМИ КОНЕЧНОГО РАЗМЕРА

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Данная работа продолжает изучение свойств функций, однородных по Эйлеру, которые можно использовать при синтезе электрических и магнитных полей электронно- и ионно-оптических систем, реализующих спектрографический режим регистрации. Рассматривается обобщение функционального уравнения общего вида для однородных функций, которое соответствует линейным функциональным соотношениям с матрицей минимального размера. В предположении о дифференцируемости рассматриваемых функций найдено общее решение построенного функционального уравнения. Полученные системы функций названы взаимно-однородными по аналогии с однородными функциями Эйлера и присоединенными однородными функциями Гельфанда.

Ключевые слова: функциональное уравнение, присоединенная однородная функция, взаимно-однородные функции, спектрограф

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Introduction

This paper continues a series of studies [1–4] on the properties of homogeneous harmonic functions and applying these functions to synthesis of electric and magnetic fields for electron and ion-optical systems using spectrographic recording [5–8].

Euler-homogeneous functions with the degree of homogeneity equal to p are real functions of several variables satisfying the following identity for any λ [9]:

$$f(\lambda x_1, \lambda x_2, \ldots, \lambda x_n) = \lambda^p f(x_1, x_2, \ldots, x_n). \quad (1)$$

Any Euler-homogeneous function has a one-to-one correspondence taking the form [9]:

$$f(x_1, x_2, \ldots, x_n) = x_1^p g(x_2/x_1, x_3/x_1, \ldots, x_n/x_1), \quad (2)$$

where $g(x_2/x_1, x_3/x_1, \ldots, x_n/x_1) = g(t_2, t_3, \ldots, t_n)$ is a real function of $(n - 1)$ variables.

Accordingly, the only homogeneous function of degree p of one variable is the power function $f(x) = \text{const}\cdot x^p$, and the only homogeneous function of degree zero of one variable is a constant.

If the function $f(x_1, x_2, \ldots, x_n)$ is differentiable, then its partial derivatives with respect to the variables $x_1, x_2, \ldots, x_n$ are homogeneous functions of degree $p - 1$ [9]. Besides, if the function $f(x_1, x_2, \ldots, x_n)$ is differentiable at any point in the space R^n, then the necessary and sufficient condition for this function to be Euler-homogeneous of degree p is that the following condition holds true at any point in the space R^n:

$$fx_1 \partial f/\partial x_1 + x_2 \partial f/\partial x_2 + \ldots + x_n \partial f/\partial x_n = pf \quad (3)$$

(the Euler theorem on homogeneous functions, also called Euler's criterion for homogeneous functions [9]).

Instead of definition (1), we can consider a functional equation of the form

$$f(\lambda x_1, \lambda x_2, \ldots, \lambda x_n) = a_0(\lambda) f(x_1, x_2, \ldots, x_n) \quad (4)$$

with the previously unknown function $a_0(\lambda)$, which, at first glance, should have more generality than condition (1). However, it soon turns out that if the function $a_0(\lambda)$ is continuous at least at one point, then the only case when Eq. (4) can have non-zero solutions that are of practical interest is a power function $a_0(\lambda) = \lambda^p$. At the same time, although Eq. (4) may have solutions different from the power function $a_0(\lambda) = \lambda^p$ and discontinuous at any point, such solutions are interesting only in the abstract mathematical sense rather than for physical applications.

Indeed, condition (4) implies that the function $a_0(\lambda)$ must satisfy the functional equation

$$\forall \lambda_1, \lambda_2 : a_0(\lambda_1 \lambda_2) = a_0(\lambda_1) a_0(\lambda_2),$$

since

$$f(\lambda_1 \lambda_2 x) = a_0(\lambda_1 \lambda_2) f(x) = a_0(\lambda_1) f(\lambda_2 x) =$$

$$= a_0(\lambda_2) f(\lambda_1 x) = a_0(\lambda_1) a_0(\lambda_2) f(x).$$

This equation is a Cauchy multiplicative functional equation. Any solution to this equation has the form of the power function $a_0(\lambda) = \lambda^p$ if the function $a_0(\lambda)$ is continuous at least at one point. The proof of this statement for differentiable functions $a_0(\lambda)$ is obtained in an elementary way after differentiating the relations

$$a_0(\lambda\mu) = a_0(\lambda) a_0(\mu)$$

with respect to μ at the point μ = 1 and the solutions of the corresponding ordinary differential equation.

Homogeneous adjoint Gelfand functions [10, 11], are a generalization of Euler-homogeneous functions; they can be defined as the solution of a semi-infinite system of functional equations:

$$f_0(\lambda x_1, \lambda x_2, \ldots) = a_0(\lambda) f_0(x_1, x_2, \ldots);$$

$$f_1(\lambda x_1, \lambda x_2, \ldots) = a_1(\lambda) f_0(x_1, x_2, \ldots) + \quad (5)$$

$$+ a_0(\lambda) f_1(x_1, x_2, \ldots);$$

$$f_2(\lambda x_1, \lambda x_2, \ldots) = a_2(\lambda) f_0(x_1, x_2, \ldots) +$$

$$+ a_1(\lambda) f_1(x_1, x_2, \ldots) + a_0(\lambda) f_2(x_1, x_2, \ldots);$$

which must be satisfied for any $\lambda > 0$; at the same time, the functions $a_k(\lambda)$ are unknown in advance.

The general solution for the system of functional equations (5), which has the form of a lower triangular matrix with identical functions $a_k(\lambda)$ along the diagonals, can be rather complex. However, only the so-called main chain of homogeneous adjoint functions can be of practical interest, for which

$$a_k(\lambda) = (1/k!)\, \lambda^p\, (\ln \lambda)^k, \quad (6)$$

$$f_k(x_1, x_2,\dots, x_n) = (1/k!)(x_1)^p (\ln x_1)^k g(x_2/x_1, x_3/x_1,\dots, x_n/x_1), \quad (7)$$

where $g(t_2, t_3,\dots,t_n)$ is an arbitrary real function of $(n-1)$ variables.

The next level of generalization are functions that must satisfy the system of functional equations

$$f_k(\lambda x_1, \lambda x_2,\dots) = \sum a_{kj}(\lambda)\, f_j(x_1, x_2,\dots), \quad (8)$$

where $k = 1, 2,\dots, m$, and the functions $a_{kj}(\lambda)$ are not known in advance.

Such functions, which we called mutually homogeneous, are considered in this paper.

The resulting formulations can have not only theoretical but also practical meaning. In particular, the principle of trajectory similarity, introduced by Golikov [5–8], holds true for Euler-homogeneous electric and magnetic potentials:

if the initial conditions of charged particles are properly scaled, then, provided that the nonrelativistic approximation holds true, article trajectories in such fields are geometrically scaled expressions.

This property allows to synthesize efficient electron and ion-optical systems, for example, such as those obtained in [12–28].

To simplify the calculations, we assume that both the functions $a_{kj}(\lambda)$, and the functions $f_k(x_1, x_2,\dots,x_n)$ are differentiable at any point. Another assumption, valid for Euler-homogeneous functions and for associated homogeneous Gelfand functions, is that imposing the condition that the functions be differentiable at all points can be considerably weakened by replacing it with the condition that the functions be continuous least at one point, producing exactly the same general formulas at the output.

Proving the corresponding theorems is beyond the scope of our study, since the requirement for differentiability at any point is always fulfilled for the scalar potentials of electric and magnetic fields used in electron and ion optics.

Matrix of minimum size

Let us consider a system of functional equations corresponding to a 2×2 matrix (8):

$$f_1(\lambda x_1, \lambda x_2,\dots) = a_{11}(\lambda) f_1(x_1, x_2,\dots) + a_{12}(\lambda) f_2(x_1, x_2,\dots), \quad (9)$$

$$f_2(\lambda x_1, \lambda x_2,\dots) = a_{21}(\lambda) f_1(x_1, x_2,\dots) + a_{22}(\lambda) f_2(x_1, x_2,\dots), \quad (10)$$

where the functions $a_{11}(\lambda)$, $a_{12}(\lambda)$, $a_{21}(\lambda)$, $a_{22}(\lambda)$ are not known in advance.

We apply one-to-one substitution of variables:

$$x = \ln x_1,\ t_2 = x_2/x_1,\ t_3 = x_3/x_1,\dots,\ t_n = x_n/x_1.$$

Substituting

$$f_1(x_1, x_2,\dots) = g_1(\ln x_1, x_2/x_1,\dots, x_n/x_1),$$

$$f_2(x_1, x_2,\dots) = g_2(\ln x_1, x_2/x_1,\dots, x_n/x_1)$$

instead of Eqs. (9), (10), we obtain the equivalent functional equations:

$$g_1(x + \ln\lambda, t_2, t_3,\dots, t_n) = a_{11}(\lambda) g_1(x, t_2, t_3,\dots, t_n) + a_{12}(\lambda)\, g_2(x, t_2, t_3,\dots, t_n), \quad (11)$$

$$g_2(x + \ln\lambda, t_2, t_3,\dots, t_n) = a_{21}(\lambda) g_1(x, t_2, t_3,\dots, t_n) + a_{22}(\lambda) g_2(x, t_2, t_3,\dots, t_n). \quad (12)$$

After differentiating Eqs. (11), (12) with respect to the variable λ at the point $\lambda = 1$, we obtain ordinary linear differential equations with constant coefficients with respect to the variable x:

$$g'_1(x,\dots) = a'_{11}(1) g_1(x,\dots) + a'_{12}(1) g_2(x,\dots), \quad (13)$$

$$g'_2(x,\dots) = a'_{21}(1) g_1(x,\dots) + a'_{22}(1) g_2(x,\dots). \quad (14)$$

The form of the analytical solution for Eqs. (13), (14) depends on the class to which the eigenvalues of the matrix $\|a'_{ij}(1)\|$ belong.

Mismatched real eigenvalues. Let the eigenvalues of the matrix (13), (14) be real and not equal to each other. The general solution for system of differential equations (13), (14) has the form

$$g_1(x, t_2, t_3,\dots, t_n) = c_{11}(t_2, t_3,\dots, t_n) \exp(p_1x) +$$

$$+ c_{12}(t_2, t_3,\dots, t_n) \exp(p_2x),$$

$$g_2(x, t_2, t_3,\dots, t_n) = c_{21}(t_2, t_3,\dots, t_n) \exp(p_2x) +$$

$$+ c_{22}(t_2, t_3,\dots, t_n) \exp(p_1x),$$

where c_{11}, c_{12}, c_{21}, c_{22} are some functions of $(n - 1)$ variables.

In this case, the functions $f_1(x_1, x_2,\dots, x_n)$ and $f_2(x_1, x_2,\dots, x_n)$ should have the form

$$f_1(x_1, x_2,\dots, x_n) = = x_1^{p1}c_{11}(x_2/x_1,\dots, x_n/x_1) + + x_1^{p2}c_{12}(x_2/x_1,\dots ,x_n/x_1), \quad (15)$$

$$f_2(x_1, x_2,\dots, x_n) = = x_1^{p1}c_{21}(x_2/x_1, x_3/x_1,\dots, x_n/x_1) + + x_1^{p2}c_{22}(x_2/x_1, x_3/x_1,\dots, x_n/x_1). \quad (16)$$

Because the functions x_1^{p1} and x_1^{p2} are linearly independent, substituting expressions (15) and (16) into conditions (9) and (10) yields the relations

$$\lambda^{p1}c_{11} = a_{11}(\lambda)c_{11} + a_{12}(\lambda)c_{21}, \quad (17)$$

$$\lambda^{p2}c_{12} = a_{11}(\lambda)c_{12} + a_{12}(\lambda)c_{22}, \quad (18)$$

$$\lambda^{p1}c_{21} = a_{21}(\lambda)c_{11} + a_{22}(\lambda)c_{21}, \quad (19)$$

$$\lambda^{p2}c_{22} = a_{21}(\lambda)c_{12} + a_{22}(\lambda)c_{22}. \quad (20)$$

Linear algebraic equations (17), (18) for unknown functions $a_{11}(\lambda)$ and $a_{12}(\lambda)$ cannot be linearly dependent (proportional to each other), except for the degenerate case

$$c_{11} = c_{12} = c_{21} = c_{22} = 0,$$

which is of no practical interest, since the functions λ^{p1} and λ^{p2} are linearly independent.

Similarly, linear algebraic equations (19), (20) for unknown functions $a_{21}(\lambda)$ and $a_{22}(\lambda)$ are also linearly independent.

Therefore, we can assume without loss of generality that

$$\Delta = c_{11}c_{22} - c_{12}c_{21} \neq 0.$$

In this case,

$$a_{11}(\lambda) = \lambda^{p1}(c_{11}c_{22}/\Delta) + \lambda^{p2}(-c_{12}c_{21}/\Delta), \quad (21)$$

$$a_{12}(\lambda) = \lambda^{p1}(-c_{11}c_{12}/\Delta) +\lambda^{p2}(c_{11}c_{12}/\Delta), \quad (22)$$

$$a_{21}(\lambda) = \lambda^{p1}(c_{21}c_{22}/\Delta) + \lambda^{p2}(-c_{21}c_{22}/\Delta), \quad (23)$$

$$a_{22}(\lambda) = \lambda^{p1}(-c_{12}c_{21}/\Delta) + \lambda^{p2}(c_{11}c_{22}/\Delta). \quad (24)$$

Since the functions $a_{11}(\lambda)$, $a_{12}(\lambda)$, $a_{21}(\lambda)$ and $a_{22}(\lambda)$ should not depend on the set of variables $x_1, x_2,\dots,x_n$, and the functions

$$c_{11}(x_2/x_1,\dots,x_n/x_1),\ c_{12}(x_2/x_1,\dots,x_n/x_1),$$

$$c_{21}(x_2/x_1,\dots,x_n/x_1),\ c_{22}(x_2/x_1,\dots,x_n/x_1)$$

should not depend on λ, the factors

$$c_{11}c_{22}/\Delta,\ c_{12}c_{21}/\Delta,\ c_{11}c_{12}/\Delta,\ c_{21}c_{22}/\Delta$$

are constants that do not depend on the given set of variables or on λ.

Therefore, the expressions

$$c_{22} : c_{12} = (c_{11}c_{22}/\Delta) : (c_{11}c_{12}/\Delta);$$

$$c_{11} : c_{21} = (c_{11}c_{22}/\Delta) : (c_{21}c_{22}/\Delta)$$

must also be constants.

As a result,

$$c_{11}(x_2/x_1, x_3/x_1, \dots, x_n/x_1) = = s_{11}h_1(x_2/x_1, x_3/x_1, \dots, x_n/x_1),$$

$$c_{12}(x_2/x_1, x_3/x_1, \dots, x_n/x_1) = = s_{12}h_2(x_2/x_1, x_3/x_1,\dots, x_n/x_1),$$

$$c_{21}(x_2/x_1, x_3/x_1,\dots, x_n/x_1) = = s_{21}h_1(x_2/x_1, x_3/x_1,\dots, x_n/x_1),$$

$$c_{22}(x_2/x_1, x_3/x_1,\dots, x_n/x_1) = = s_{22}h_2(x_2/x_1, x_3/x_1,\dots, x_n/x_1),$$

where the values s_{11}, s_{12}, s_{21}, s_{22} are arbitrary constants; $h_1(t_2, t_3,...,t_n)$, $h_1(t_2, t_3,...,t_n)$ are arbitrary functions of $(n - 1)$ variables.

The final form that the general solution for functional equations (9) and (10) takes is

$$a_{11}(\lambda) = \lambda^{p1} + (\lambda^{p2} - \lambda^{p1})(-s_{12}s_{21}/\Delta^*), \quad (25)$$

$$a_{12}(\lambda) = (\lambda^{p2} - \lambda^{p1})(s_{11}s_{12}/\Delta^*), \quad (26)$$

$$a_{21}(\lambda) = (\lambda^{p2} - \lambda^{p1})(-s_{21}s_{22}/\Delta^*), \quad (27)$$

$$a_{22}(\lambda) = \lambda^{p2} + (\lambda^{p2} - \lambda^{p1})(s_{12}s_{21}/\Delta^*), \quad (28)$$

$$f_1(x_1, x_2,\ldots, x_n) = = x_1^{p1}s_{11}h_1(x_2/x_1, x_3/x_1,\ldots,x_n/x_1) + + x_1^{p2}s_{12}h_2(x_2/x_1, x_3/x_1, \ldots,x_n/x_1), \quad (29)$$

$$f_2(x_1, x_2,\ldots, x_n) = x_1^{p1} s_{21}h_1(x_2/x_1, x_3/x_1,\ldots, x_n/x_1) + + x_1^{p2}s_{22}h_2(x_2/x_1, x_3/x_1,\ldots, x_n/x_1), \quad (30)$$

where

$$\Delta^* = s_{11}s_{22} - s_{12}s_{21} \neq 0,$$

and s_{11}, s_{12}, s_{21}, s_{22} are arbitrary constants; $h_1(t_2, t_3,...,t_n)$ and $h_2(t_2, t_3,...,t_n)$ are arbitrary functions of $(n - 1)$ variables.

In the general case, some of the constants in Eqs. (25)–(30) are redundant because, for example, the constant s_{11} can be combined with the function

$$h_1(x_2/x_1, x_3/x_1,\ldots, x_n/x_1),$$

and the constant s_{22} with the function

$$h_2(x_2/x_1, x_3/x_1,\ldots, x_n/x_1),$$

however, then the cases $s_{11} = 0$ or $s_{22} = 0$ have to be considered separately. In particular, we can assume without loss of generality that $s_{11} = s_{22} = 1$ in the general formulas, and regard cases when $s_{11} = s_{22} = 0$ or $s_{11} = 0$, $s_{22} = 1$ as degenerate.

Notably, Eqs. (25)–(30) hold true even with $p_1 = p_2 = p$, when they take the form

$$a_{11}(\lambda) = \lambda^p, \; a_{12}(\lambda) = 0,$$

$$a_{21}(\lambda) = 0, \; a_{22}(\lambda) = \lambda^p,$$

$$f_1(x_1, x_2,\ldots, x_n) = = x_1^p h_1(x_2/x_1, x_3/x_1,\ldots, x_n/x_1),$$

$$f_2(x_1, x_2,\ldots, x_n) = = x_1^p h_2(x_2/x_1, x_3/x_1,\ldots, x_n/x_1),$$

i.e., the solution splits into two independent homogeneous functions of the same degree in this case.

Equal real eigenvalues. Let the eigenvalues of matrix (13), (14) be real and equal to each other. The general solution for system of differential equations (13), (14) has the form

$$g_1(x, t_2, t_3,\ldots, t_n) = c_{11}(t_2, t_3,\ldots, t_n) \exp(px) + + c_{12}(t_2, t_3,\ldots, t_n)\, x \exp(px);$$

$$g_2(x, t_2, t_3,\ldots, t_n) = c_{21}(t_2, t_3,\ldots, t_n) \exp(px) + + c_{22}(t_2, t_3,\ldots, t_n)\, x \exp(px),$$

where c_{11}, c_{12}, c_{21}, c_{22} are some functions of $(n - 1)$ variables.

In this case, the functions

$$f_1(x_1, x_2,\ldots, x_n), f_2(x_1, x_2,\ldots, x_n)$$

should have the form

$$f_1(x_1, x_2,\ldots, x_n) = = x_1^p c_{11}(x_2/x_1, x_3/x_1,\ldots, x_n/x_1) + + x_1^p(\ln x_1)c_{12}(x_2/x_1, x_3/x_1,\ldots, x_n/x_1), \quad (31)$$

$$f_2(x_1, x_2,\ldots, x_n) = = x_1^p c_{21}(x_2/x_1, x_3/x_1,\ldots, x_n/x_1) + + x_1^p (\ln x_1)c_{22}(x_2/x_1, x_3/x_1,\ldots, x_n/x_1). \quad (32)$$

Because the functions x_1^p and x_1^p (ln x_1) are linearly independent, substituting expressions (31) and (32) into conditions (9) and (10) yields the relations

$$a_{11}(\lambda)c_{11} + a_{12}(\lambda)c_{21} = \lambda^p c_{11}, \quad (33)$$

$$a_{11}(\lambda)c_{12} + a_{12}(\lambda)c_{22} = \lambda^p(\ln\lambda)c_{12}, \quad (34)$$

$$a_{21}(\lambda)c_{11} + a_{22}(\lambda)c_{21} = \lambda^p c_{21}, \quad (35)$$

$$a_{21}(\lambda)c_{12} + a_{22}(\lambda)c_{22} = \lambda^p(\ln\lambda)c_{22}. \quad (36)$$

Since the functions λ^p и λ^p (lnλ) are linearly independent, linear algebraic equations (33), (34) for unknown functions $a_{11}(\lambda)$ и $a_{12}(\lambda)$, as well as linear algebraic equations (35), (36) for unknown functions $a_{21}(\lambda)$ and

$a_{22}(\lambda)$ cannot be linearly dependent (proportional to each other), with the exception of the degenerate case $c_{12} = c_{22} = 0$, considered separately.

Let

$$\Delta = c_{11}\, c_{22} - c_{12}\, c_{21} \neq 0.$$

In this case,

$$a_{11}(\lambda) = \lambda^p(1 + (1 - \ln\lambda)(c_{12}c_{21}/\Delta)), \quad (37)$$

$$a_{12}(\lambda) = \lambda^p(1 - \ln\lambda)(-c_{11}c_{12}/\Delta), \quad (38)$$

$$a_{21}(\lambda) = \lambda^p(1 - \ln\lambda)(c_{21}c_{22}/\Delta), \quad (39)$$

$$a_{22}(\lambda) = \lambda^p(1 + (1 - \ln\lambda)\,(-c_{11}c_{22}/\Delta)). \quad (40)$$

Since the functions $a_{11}(\lambda)$, $a_{12}(\lambda)$, $a_{21}(\lambda)$ and $a_{22}(\lambda)$ should not depend on the set of variables $x_1, x_2,\ldots,x_n$, and functions

$$c_{11}(x_2/x_1, x_3/x_1,\ldots, x_n/x_1),$$

$$c_{12}(x_2/x_1, x_3/x_1,\ldots, x_n/x_1),$$

$$c_{21}(x_2/x_1, x_3/x_1,\ldots, x_n/x_1),$$

$$c_{22}(x_2/x_1, x_3/x_1,\ldots, x_n/x_1)$$

should not depend on λ, then the factors $c_{12}c_{21}/\Delta$, $c_{11}c_{12}/\Delta$, $c_{21}c_{22}/\Delta$, $c_{11}c_{22}/\Delta$ are constants that do not depend on either the given set of variables or on λ.

Therefore, the expressions $c_{21} : c_{11} = (c_{12}c_{21}/\Delta) : (c_{11}c_{12}/\Delta)$ and $c_{12} : c_{22} = (c_{11}c_{12}/\Delta) : (c_{11}c_{22}/\Delta)$ are also constants, so that

$$c_{11}(x_2/x_1, x_3/x_1,\ldots, x_n/x_1) = s_{11}$$
$$h_1(x_2/x_1, x_3/x_1,\ldots, x_n/x_1),$$

$$c_{21}(x_2/x_1, x_3/x_1,\ldots, x_n/x_1) = s_{21}$$
$$h_1(x_2/x_1, x_3/x_1,\ldots, x_n/x_1),$$

$$c_{12}(x_2/x_1, x_3/x_1,\ldots, x_n/x_1) = s_{12}$$
$$h_2(x_2/x_1, x_3/x_1,\ldots, x_n/x_1),$$

$$c_{22}(x_2/x_1, x_3/x_1,\ldots, x_n/x_1) = s$$
$$_{22}\, h_2(x_2/x_1, x_3/x_1,\ldots, x_n/x_1),$$

where s_{11}, s_{12}, s_{21} and s_{22} are constants that are not simultaneously equal to zero; $h_1(x_2/x_1, x_3/x_1,\ldots, x_n/x_1)$ and $h_2(x_2/x_1, x_3/x_1,\ldots, x_n/x_1)$ are some functions of $(n - 1)$ variables.

The solution takes the final form

$$a_{11}(\lambda) = \lambda^p(1 + (1 - \ln\lambda)(s_{12}s_{22}/\Delta^*)), \quad (41)$$

$$a_{12}(\lambda) = \lambda^p(1 - \ln\lambda)(-s_{11}s_{12}/\Delta^*), \quad (42)$$

$$a_{21}(\lambda) = \lambda^p(1 - \ln\lambda)(s_{21}s_{22}/\Delta^*), \quad (43)$$

$$a_{22}(\lambda) = \lambda^p(1 + (1 - \ln\lambda)(-s_{11}s_{22}/\Delta^*)), \quad (44)$$

$$f_1(x_1, x_2,\ldots, x_n) =$$
$$= x_1^p s_{11} h_1(x_2/x_1, x_3/x_1,\ldots, x_n/x_1) + \quad (45)$$
$$+ x_1^p(\ln x_1) s_{12} h_2(x_2/x_1, x_3/x_1,\ldots, x_n/x_1),$$

$$f_2(x_1, x_2,\ldots, x_n) =$$
$$= x_1^p s_{21} h_1(x_2/x_1, x_3/x_1,\ldots, x_n/x_1) + \quad (46)$$
$$+ x_1^p(\ln x_1) s_{22} h_2(x_2/x_1, x_3/x_1,\ldots, x_n/x_1),$$

where $\Delta^* = s_{11}s_{22} - s_{12}s_{21} \neq 0$, s_{11}, s_{12}, s_{21}, s_{22} are arbitrary constants, and $h_1(t_2, t_3,\ldots,t_n)$ and $h_2(t_2, t_3,\ldots,t_n)$ are arbitrary functions of $(n - 1)$ variables.

Complex conjugate eigenvalues. Let the eigenvalues of matrix (13), (14) be conjugate complex numbers taking the form $p \pm i\omega$.

The general solution for system of differential equations (13), (14) has the form

$$g_1(x, t_2, t_3,\ldots, t_n) =$$
$$= c_{11}(t_2, t_3,\ldots, t_n)\cos(\omega x)\exp(px) +$$
$$+ c_{12}(t_2, t_3,\ldots, t_n)\sin(\omega x)\exp(px);$$

$$g_2(x, t_2, t_3,\ldots, t_n) =$$
$$= c_{21}(t_2, t_3,\ldots, t_n)\cos(\omega x)\exp(px) +$$
$$+ c_{22}(t_2, t_3,\ldots, t_n)\sin(\omega x)\exp(px),$$

where c_{11}, c_{12}, c_{21}, c_{22} are some functions of $(n - 1)$ variables.

In this case, the functions f_1 и f_2 should have the following form:

$$f_1(x_1, x_2,\ldots, x_n) =$$
$$= x_1^p c_{11}(x_2/x_1, x_3/x_1,\ldots, x_n/x_1)\cos(\omega\ln x_1) + \quad (47)$$
$$+ x_1^p c_{12}(x_2/x_1, x_3/x_1,\ldots, x_n/x_1)\sin(\omega\ln x_1);$$

$$f_2(x_1, x_2,\ldots, x_n) =$$
$$= x_1^p c_{21}(x_2/x_1, x_3/x_1,\ldots, x_n/x_1)\cos(\omega\ln x_1) + \quad (48)$$
$$+ x_1^p c_{22}(x_2/x_1, x_3/x_1,\ldots, x_n/x_1)\sin(\omega\ln x_1).$$

Because the functions $x_1^p\cos(\omega\ln x_1)$ and $x_1^p\sin(\omega\ln x_1)$ are linearly independent, substituting expressions (47) and (48) into conditions (9) and (10) yields the relations

$$c_{11}a_{11}(\lambda) + c_{21}a_{12}(\lambda) = \lambda^p c_{11}\cos(\omega\ln\lambda) + \lambda^p c_{12}\sin(\omega ln\lambda), \quad (49)$$

$$c_{12}a_{11}(\lambda) + c_{22}a_{12}(\lambda) = \lambda^p c_{12}\cos(\omega ln\lambda) - \lambda^p c_{11}\sin(\omega ln\lambda), \quad (50)$$

$$c_{11}a_{21}(\lambda) + c_{21}a_{22}(\lambda) = \lambda^p c_{21}\cos(\omega\ln\lambda) + \lambda^p c_{22}\sin(\omega\ln\lambda), \quad (51)$$

$$c_{12}a_{21}(\lambda) + c_{22}a_{22}(\lambda) = \lambda^p c_{22}\cos(\omega\ln\lambda) - \lambda^p c_{21}\sin(\omega\ln\lambda). \quad (52)$$

Linear algebraic equations (49), (50) for unknown functions $a_{11}(\lambda)a_{12}(\lambda)$ and linear algebraic equations (51), (52) for unknown functions $a_{21}(\lambda)a_{22}(\lambda)$ cannot be linearly dependent, except the degenerate case

$$c_{11} = c_{12} = c_{21} = c_{22} = 0,$$

which is of no practical interest.

Indeed, the functions $\lambda^p\cos(\omega\ln\lambda)$ and $\lambda^p\sin(\omega\ln\lambda)$ are linearly independent, while the proportionality relations

$$c_{11} : c_{12} = c_{12} : (-c_{11}),$$

$$c_{21} : c_{22} = c_{22} : (-c_{21})$$

for the right-hand sides of Eqs. (49)–(52) cannot be satisfied with nonzero values of c_{11}, c_{12}, c_{21}, c_{22}.

Therefore, without loss of generality, we can assume that

$$\Delta = c_{11}c_{22} - c_{12}c_{21} \neq 0.$$

In this case,

$$a_{11}(\lambda) = \lambda^p\cos(\omega\ln\lambda) + \lambda^p\sin(\omega\ln\lambda)((c_{11}c_{21} + c_{12}c_{22})/\Delta), \quad (53)$$

$$a_{12}(\lambda) = -\lambda^p\sin(\omega\ln\lambda)((c_{11}^2 + c_{12}^2)/\Delta), \quad (54)$$

$$a_{21}(\lambda) = +\lambda^p\sin(\omega\ln\lambda)((c_{21}^2 + c_{22}^2)/\Delta), \quad (55)$$

$$a_{22}(\lambda) = \lambda^p\cos(\omega\ln\lambda) - \lambda^p\sin(\omega\ln\lambda)((c_{11}c_{21} + c_{12}c_{22})/\Delta). \quad (56)$$

Since the functions $a_{11}(\lambda)$, $a_{12}(\lambda)$, $a_{21}(\lambda)$ and $a_{22}(\lambda)$ should not depend on the set of variables $x_1, x_2,\ldots,x_n$, and the functions

$$c_{11}(x_2/x_1, x_3/x_1,\ldots, x_n/x_1),$$

$$c_{12}(x_2/x_1, x_3/x_1,\ldots, x_n/x_1),$$

$$c_{21}(x_2/x_1, x_3/x_1,\ldots, x_n/x_1),$$

$$c_{22}(x_2/x_1, x_3/x_1,\ldots, x_n/x_1)$$

should not depend on λ, the factors

$$(c_{11}c_{21}+c_{12}c_{22})/\Delta,\ (c_{11}^2+c_{12}^2)/\Delta,$$

$$(c_{21}^2+c_{22}^2)/\Delta$$

are constants that do not depend on the given set of variables or on λ.

Substituting

$$c_{11}(x_2/x_1, x_3/x_1,\ldots) = h_a(x_2/x_1, x_3/x_1,\ldots)\cos h_b(x_2/x_1, x_3/x_1,\ldots),$$

$$c_{12}(x_2/x_1, x_3/x_1,\ldots) = h_a(x_2/x_1, x_3/x_1,\ldots)\sin h_b(x_2/x_1, x_3/x_1,\ldots),$$

$$c_{21}(x_2/x_1, x_3/x_1,\ldots) = h_c(x_2/x_1, x_3/x_1,\ldots)\sin h_d(x_2/x_1, x_3/x_1,\ldots),$$

$$c_{22}(x_2/x_1, x_3/x_1,\ldots) = h_c(x_2/x_1, x_3/x_1,\ldots)\cos h_d(x_2/x_1, x_3/x_1,\ldots),$$

we obtain that the constants should be

$$\mathrm{tg}\,(h_b(x_2/x_1, x_3/x_1,\ldots) + h_d(x_2/x_1, x_3/x_1,\ldots));$$

$$h_a(x_2/x_1, x_3/x_1,\ldots)/h_c(x_2/x_1, x_3/x_1,\ldots).$$

Therefore, after substitutions

$$h_a(x_2/x_1, x_3/x_1,\ldots) = s_a\,h(x_2/x_1, x_3/x_1,\ldots),$$

$$h_c(x_2/x_1, x_3/x_1,\ldots) = s_c\,h(x_2/x_1, x_3/x_1,\ldots),$$

$$h_b(x_2/x_1, x_3/x_1,\ldots) = f(x_2/x_1, x_3/x_1,\ldots) + s_b,$$

$$h_d(x_2/x_1, x_3/x_1,\ldots) = -f(x_2/x_1, x_3/x_1,\ldots) + s_d,$$

where s_a, s_c, s_b, s_d are constants; $h(x_2/x_1, x_3/x_1,\ldots)f(x_2/x_1, x_3/x_1,\ldots)$ are auxiliary functions, and after some additional equivalent transformations, we obtain the formulas

$$c_{11}(x_2/x_1, x_3/x_1,\ldots) =$$
$$+ s_{11}h(x_2/x_1, x_3/x_1,\ldots)\cos f(x_2/x_1, x_3/x_1,\ldots) -$$
$$- s_{12}h(x_2/x_1, x_3/x_1,\ldots)\sin f(x_2/x_1, x_3/x_1,\ldots);$$
$$c_{12}(x_2/x_1, x_3/x_1,\ldots) =$$
$$= + s_{11}h(x_2/x_1, x_3/x_1,\ldots)\sin f(x_2/x_1, x_3/x_1,\ldots) +$$
$$+ s_{12}h(x_2/x_1, x_3/x_1,\ldots)\cos f(x_2/x_1, x_3/x_1,\ldots);$$
$$c_{21}(x_2/x_1, x_3/x_1,\ldots) =$$
$$= - s_{22}h(x_2/x_1, x_3/x_1,\ldots)\sin f(x_2/x_1, x_3/x_1,\ldots) +$$
$$+ s_{21}h(x_2/x_1, x_3/x_1,\ldots)\cos f(x_2/x_1, x_3/x_1,\ldots);$$
$$c_{22}(x_2/x_1, x_3/x_1,\ldots) =$$
$$= + s_{22}h(x_2/x_1, x_3/x_1,\ldots)\cos f(x_2/x_1, x_3/x_1,\ldots) +$$
$$+ s_{21}h(x_2/x_1, x_3/x_1,\ldots)\sin f(x_2/x_1, x_3/x_1,\ldots),$$

where s_{11}, s_{12}, s_{21}, s_{22} are constants that do not depend on the given set of variables or on λ.

Such a choice of parameterization for c_{11}, c_{12}, c_{21}, c_{22} is redundant (obviously) because we can establish, for example, $s_a = 1$ and $s_b = 0$ practically without loss of generality, which means that $s_{11} = 1$ and $s_{12} = 0$.

Besides, it is convenient to substitute

$$h(x_2/x_1, x_3/x_1,\ldots)\cos f(x_2/x_1, x_3/x_1,\ldots)$$

with $h_1(x_2/x_1, x_3/x_1,...)$, and

$$h(x_2/x_1, x_3/x_1,\ldots)\sin f(x_2/x_1, x_3/x_1,\ldots)$$

with $h_2(x_2/x_1, x_3/x_1,...)$.

The final form that the general solution for functional equations (9) and (10) takes is

$$a_{11}(\lambda) = \lambda^p\cos(\omega\ln\lambda) + \lambda^p\sin(\omega\ln\lambda) \times \times ((s_{11}s_{21} + s_{12}s_{22})/\Delta^*), \quad (57)$$

$$a_{12}(\lambda) = - \lambda^p\sin(\omega\ln\lambda)((s_{11}^2 + s_{12}^2)/\Delta^*), \quad (58)$$

$$a_{21}(\lambda) = + \lambda^p\sin(\omega\ln\lambda)((s_{21}^2 + s_{22}^2)/\Delta^*), \quad (59)$$

$$a_{22}(\lambda) = \lambda^p\cos(\omega\ln\lambda) - \lambda^p\sin(\omega\ln\lambda) \times \times ((s_{11}s_{21} + s_{12}s_{22})/\Delta^*), \quad (60)$$

$$f_1(x_1, x_2,\ldots, x_n) = x_1^p(s_{11}\cos(\omega\ln x_1) + + s_{12}\, x_1^p\sin(\omega\ln x_1))h_1(x_2/x_1, x_3/x_1,\ldots) + + x_1^p(- s_{12}\cos(\omega\ln x_1) + + s_{11}\sin(\omega\ln x_1))\, h_2(x_2/x_1, x_3/x_1,\ldots), \quad (61)$$

$$f_2(x_1, x_2,\ldots, x_n) = x_1^p(s_{21}\cos(\omega\ln x_1) + + s_{22}\, x_1^p\sin(\omega\ln x_1))h_1(x_2/x_1, x_3/x_1,\ldots) + + x_1^p\,(- s_{22}\cos(\omega\ln x_1) + + s_{21}\sin(\omega\ln x_1))\, h_2(x_2/x_1, x_3/x_1,\ldots), \quad (62)$$

where

$$\Delta^* = s_{11}s_{22} - s_{12}s_{21} \neq 0,$$

and s_{11}, s_{12}, s_{21}, s_{22} are arbitrary constants (partially redundant); $h_1(t_2, t_3,...,t_n)$, $h_2(t_2, t_3,...,t_n)$ are arbitrary functions of $(n - 1)$ variables.

Further steps

We can analyze other systems of functional equations of the form (8) with matrices of finite size by a similar scheme. However, complex formulas with many variant branches appear as a result of analysis; in our opinion, they have no particular practical meaning.

Taking into account the analysis given in this article for differentiable functions, all solutions of functional equations of the form (8) are linear combinations of functions taking the form

$$f_{k,p}(x_1, x_2,\ldots, x_n) = = x_1^p(\ln x_1)^k h(x_2/x_1, x_3/x_1,\ldots, x_n/x_1), \quad (63)$$

which correspond to real eigenvalues p with the multiplicity k, and functions taking the form

$$f^{(c)}_{k,p}(x_1, x_2,\ldots, x_n) = x_1^p(\ln x_1)^k\cos(\omega\ln x_1) \times \times h(x_2/x_1, x_3/x_1,\ldots, x_n/x_1), \quad (64)$$

$$f^{(s)}_{k,p}(x_1, x_2,\ldots, x_n) = x_1^p(\ln x_1)^k\sin(\omega\ln x_1) \times \times h(x_2/x_1, x_3/x_1,\ldots, x_n/x_1), \quad (65)$$

which correspond to complex conjugate eigenvalues of the form $p \pm i\omega$ of multiplicity k, where $h(t_2, t_3,...,t_n)$ are some functions of $(n - 1)$ variables.

Regarding the theory on mutually homogeneous functions, we believe that it is sufficient to analyze systems of functional relations corresponding to isolated fundamental chains of functions taking the form (63) and (64), (65), instead of analyzing systems of functional relations with a general form.

We plan to carry out further investigations analyzing systems of mutually homogeneous functions with infinite chains of functional equations of the form (8).

The calculations in this paper were carried out using the Wolfram Mathematica software [29].

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