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POSITIVE COLUMN OF A DIRECT CURRENT DISCHARGE IN LASER TUBES OF VARIABLE DIAMETER

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A positive column of a direct current discharge in monoatomic gas is considered and expressions are obtained that relate the external parameters of the column (varying radius of the discharge channel, gas pressure and discharge current) to the "internal" characteristics (concentration of charged particles, electron temperature, and longitudinal electric field strength).

Keywords: positive column plasma characteristic, variable diameter laser tube, active element geometry

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ПОЛОЖИТЕЛЬНЫЙ СТОЛБ РАЗРЯДА ПОСТОЯННОГО ТОКА

В ЛАЗЕРНЫХ ТРУБКАХ ПЕРЕМЕННОГО ДИАМЕТРА

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Рассмотрены процессы в положительном столбе разряда постоянного тока в газе и получены выражения, связывающие внешние параметры столба (меняющийся радиус разрядного канала, давление напуска газа и разрядный ток) с «внутренними» характеристиками (концентрация заряженных частиц, электронная температура, напряженность «продольного» электрического поля).

Ключевые слова: характеристика плазмы положительного столба, лазерная трубка переменного диаметра, геометрия активного элемента

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Gas-discharge lasers such as, for example, helium-neon (He-Ne) or helium-cadmium (He-Cd), are one of the most common types, using a positive column (PC) of low-pressure direct-current glow discharge in cylindrical tubes. The laser's gain medium is placed in an optical cavity with sphere-plane geometry. The shape of the caustic in such a cavity is considerably different from cylindrical. Because of this, some of the excited atoms produced in the cylindrical discharge do not contribute to the 'effective' gain determining the laser's output power. Conical tubes were first considered in 1969 as a means for improving the efficiency of the gain medium and increasing its mode volume in gas lasers [1, 2]. This model was supported by calculations of the gain [3] and confirmed experimentally [4].

The 'geometric' part of the gain k were based on the formula

$$k = \frac{1}{S}\int_V k_0 \cdot f_s dV,$$

where k_0 is the unsaturated gain on the tube axis, f_S is the gain distribution function over the column cross-section, S is its cross-sectional area.

It was assumed that the function f_S is similar to the distribution of the concentration of excited atoms in the discharge [5, 6]. This formulation implies that the quantities k_0, f_S and S are independent of the longitudinal coordinate z of the column. This is questionable to say the least for discharge in a conical tube, since the area S and such important characteristics of the PC as electron concentration and temperature, determining the population inversion, change with the changing radius of the discharge channel.

An earlier study [7] considered the reaction of the parameters of the positive column to an abrupt change in the radius of the discharge tube. However, there is practically no data in literature for tubes with a 'smoothly changing' radius of the discharge channel, especially in GDLs. Our study is dedicated to this problem.

Let us consider a direct current PC of length l in monatomic gas. Gas inlet pressure p does not exceed 10 mm Hg, discharge current I lies in the range of 10–100 mA. The radius R of the discharge channel is a smooth function of the coordinate z ($0 \le z \le l$):

$$|dR/dz| << 1.$$

The axis z is directed along the axis of the discharge tube:

$$R = R(z) = R_0 f_R(z),$$

where R_0 is the radius of the channel R at the point $z = 0$; f_R $(z = 0) = 1$;

$$1 \text{ mm} \le R_0 \le 5 \text{ mm}.$$

We assume that the given positive column under these discharge conditions is three-component plasma consisting of neutral atoms of the same kind, singly charged positive ions, and electrons. The concentrations of these particles are denoted, respectively, as n_a, n_i, n_e, and their masses as m_a, m_i, m_e; $m_a = m_i$.

The ion mean free path λ_i is much smaller than the tube radius: $\lambda_i << R$. Therefore, the processes in the discharge can be considered within Schottky's diffusion theory of the positive column.

We assume that the column plasma is quasi-neutral in these discharge conditions, i.e., $n_e \approx n_i = n$, and is weakly ionized:

$$\nu_{ee}, \nu_{ei}, \nu_{ii} << \nu_{ea}, \nu_{ia},$$

i.e., the frequencies with which charged particles collide with each other, ν_{ee}, ν_{ei}, ν_{ii}, are much lower than the frequencies of electron-atom (ν_{ea}) and ion-atom (ν_{ia}) collisions.

Such processes as stepwise ionization, bulk recombination, and electron attachment are unlikely under this assumption.

Next let us assume the energy distributions of electrons, ions, and neutral atoms to be Maxwellian, with temperatures $T_\alpha \equiv T_e$, T_i, T_a, respectively. We also assume that

particle temperatures are related as $T_e >> T_i \approx T_a$;

the electron temperature is uniform over the column cross-section but is a function of the coordinate z: T_e = const (r,θ), $T_e = T_e(z)$;

the ion temperature is distributed uniformly both over the column cross-section and along the longitudinal coordinate: T_e = const (r,θ,z).

The distributions of gas pressure in the column, equal to the inlet pressure p, gas temperature, and, consequently, the concentration of neutral particles $n_a = p/kT_a$ are assumed to be uniform both over the cross-section of the column and along its length: T_a, n_a = const (r,θ,z).

The concentrations of charged particles are azimuthally uniform but are functions of the radial and longitudinal coordinates:

$$n_e = \text{const}(\theta),\ n_e = n_e(r,z),$$

$$n_i = \text{const}(\theta),\ n_i = n_i(r,z).$$

The electric field **E** is also azimuthally uniform and is a function of the radial and longitudinal coordinates:

$$\mathbf{E} = \mathrm{const}(\theta),\ \mathbf{E} = \mathbf{E}(r,z).$$

We assume that the main mechanism for producing charged particles is direct ionization by electron impact from the ground state of atoms. The ionization frequency is $\nu_i = n_a\langle\sigma_{0i},\mathrm{v}_e\rangle$, where $\langle\sigma_{0i},\mathrm{v}_e\rangle$ is the product $\sigma_{0i}\mathrm{v}_e$, averaged over the electron distribution function $f_e(\mathrm{v}_e)$; σ_{0i} is the cross-section of this ionization process; v_e is the electron velocity.

We assume that the main mechanism for the decay of charged particles is their diffusive escape to the walls of the discharge tube.

The current density $\mathbf{j} = \mathbf{j}(r,z)$ is equal to the difference in the density of ion fluxes $\mathbf{\Gamma}_i = n_i\mathbf{u}_i$ and electron fluxes $\mathbf{\Gamma}_e = n_e\mathbf{u}_e$:

$$\mathbf{j} = e(\mathbf{\Gamma}_i - \mathbf{\Gamma}_e) = e\,(n_i\mathbf{u}_i - n_e\mathbf{u}_e) = en(\mathbf{u}_i - \mathbf{u}_e). \quad (1)$$

The discharge current I is determined by the zth component of the current density $\mathbf{j}_z$ associated with the drift of charged particles in the field $\mathbf{E}_z$ (the longitudinal potential gradient in the column) depending on the longitudinal coordinate:

$$\mathbf{E}_z(z) = \mathbf{E}_{0_z}\mathrm{f}_{E_z}(z),$$

where $\mathbf{E}_{0_z} = \mathbf{E}_z(z = 0)$; $f_{E_z}(z = 0) = 1$.

We assume that all energy is supplied to the PC from electrons accelerated in this electric field $\mathbf{E}_z$.

The following problem is formulated:

obtain relatively simple expressions linking easily controlled external parameters of the column (discharge current I, radius R(z) of the discharge channel and gas inlet pressure p) with its main internal characteristics (charged particle concentration $n_e(z)$*, electron temperature* $T_e(z)$ *and strength of the longitudinal electric field* $\mathbf{E}_z(z)$*).*

Based on the methods developed in [7, 8, 12], we use the following equations to solve this problem.

Equations of motion of charged particles, which, provided that

$$\nu_{ee},\ \nu_{ei},\ \nu_{ii} << \nu_{ea},\ \nu_{ia},$$

and neglecting the thermal power, have the form:

$$-e\mathbf{E} - \frac{\nabla\left(n_e kT_e\right)}{n_e} - \mu_{ea}\nu_{ea}\mathbf{u}_e = 0$$

for electrons;

$$e\mathbf{E} - \frac{\nabla\left(n_i kT_i\right)}{n_i} - \mu_{ia}\nu_{ia}u_i = 0$$

for ions.

Here μ_{ea}, μ_{ia} are the normalized masses of electrons and ions, respectively.

Equations of balance of charged particles:

$$\frac{\partial n_e}{\partial t} + \nabla\left(n_e\mathbf{u}_e\right) = \frac{\delta n_e}{\delta t}$$

for electrons;

$$\frac{\partial n_i}{\partial t} + \nabla\left(n_i\mathbf{u}_i\right) = \frac{\delta n_i}{\delta t}$$

for ions.

The equation of electron energy balance:

$$IE_z = P_\nu + P_w,$$

where IE_z is the power expended by the longitudinal electric field to heat the electrons and maintain the total energy balance in the column; P_ν is the power lost by electrons in elastic collisions with atoms (gas heating); P_w is the power transferred to the wall by ions.

The power lost by electrons in inelastic collisions with atoms is not taken into account in the balance equation.

Let us consider these equations in more detail.

The expressions for the directed velocities of electrons and ions for a direct-current discharge are written as follows assuming that plasma is quasi-neutral and that ion temperatures are independent of the coordinates:

$$\mathbf{u}_e = -b_e\mathbf{E} - D_e\left(\frac{\nabla n}{n} + \frac{\nabla T_e}{T_e}\right); \quad (2)$$

$$\mathbf{u}_i = +b_i\mathbf{E} - D_i\frac{\nabla n}{n}. \quad (3)$$

Here $b_e = e/\mu_{ea}\nu_{ea}$, $b_i = e/\mu_{ia}\nu_{ia}$ are the mobilities of electrons and ions; $D_e = kT_e/\mu_{ea}\nu_{ea} = b_e kT_e/e$, $D_i = kT_i/\mu_{ia}\nu_{ia} = b_i kT_i/I$ are the diffusion coefficients of electrons and ions, respectively; ν_{ea} is the frequency of elastic electron-atom collisions, equal to

$$\nu_{ea} = n_a\left\langle\sigma_{ea}v_e\right\rangle,$$

where $\langle\sigma_{ea},\nu_e\rangle$ is the product $\sigma_{ea}\nu_e$, averaged over the electron distribution function $f_e(\nu_e)$; σ_{ea} is the cross-section of elastic electron-atom collisions, ν_e is the electron velocity.

In general, the expression $\langle\sigma_{ea},\nu_e\rangle$ is also a function of the longitudinal coordinate z, because $T_e = T_e(z)$. We also assume that

$$\langle \sigma_{ea} v_e \rangle = \text{const}(z),$$

which is true to a first approximation, at least for discharge in helium, since $\sigma_{ea}\nu_{ea} \approx \text{const}(T_e)$ in the variation range of electron energy characteristic for these discharge conditions [11].

The frequency of elastic ion-atom collisions has the form

$$\nu_{ia}\nu_{ia} = n_a \langle \sigma_{ia} v_{ia} \rangle,$$

where the quantity $\langle \sigma_{ia}, \nu_{ia} \rangle$ can be represented for the case of weakly ionized plasma as

$$\langle \sigma_{ia} v_{ia} \rangle \approx \sigma_{ia} \langle v_{T_a} \rangle = \sigma_{ia} \sqrt{3\frac{kT_a}{m_a}} = \text{const}(z).$$

The current density in DC plasma is then written as

$$\mathbf{j} = en\left[b_i\mathbf{E} - D_i\frac{\nabla n}{n} + b_e\mathbf{E} + D_e\left(\frac{\nabla n}{n} + \frac{\nabla kT_e}{kT_e}\right)\right]. \tag{4}$$

We then obtain the expression for the electric field $\mathbf{E}$:

$$\mathbf{E} = \frac{\mathbf{j}}{en(b_i + b_e)} - \left(\frac{D_e - D_i}{b_i + b_e}\cdot\frac{\nabla n}{n} + \frac{D_e}{b_i + b_e}\cdot\frac{\nabla kT_e}{kT_e}\right) = \mathbf{E}_{con} + \mathbf{E}_{sc}.$$

In other words, the electric field $\mathbf{E}$ in column plasma is determined by two terms. The first one is the electric field of 'conductivity' $\mathbf{E}_{con}$, that is, the 'external' field generating the current flux $\mathbf{j}$ in a medium with the conductivity σ:

$$E_{con} = \frac{\mathbf{j}}{en(b_i + b_e)} = \frac{\mathbf{j}}{\sigma}.$$

The second term is the 'space charge' field $\mathbf{E}_{sc}$:

$$\mathbf{E}_{sc} = -\left(\frac{D_e - D_i}{b_i + b_e}\cdot\frac{\nabla n}{n} + \frac{D_e}{b_i + b_e}\cdot\frac{\nabla kT_e}{kT_e}\right) = \left(\mathbf{E}_{\hat{\imath}\varsigma_{\nabla n}} + \mathbf{E}_{\hat{\imath}\varsigma_{\nabla T_e}}\right),$$

where

$$\mathbf{E}_{sc_{\nabla n}} = -\frac{D_e - D_i}{b_i + b_e}\cdot\frac{\nabla n}{n},$$

$$\mathbf{E}_{sc_{\nabla T_e}} = -\frac{D_e}{b_i + b_e}\cdot\frac{\nabla kT_e}{kT_e}.$$

The expression for the current density can then be written as

$$\mathbf{j} = en\left[(b_i + b_e)(\mathbf{E}_{con} + \mathbf{E}_{sc}) + (D_e - D_i)\frac{\nabla n_e}{n_e} + D_e\frac{\nabla kT_e}{kT_e}\right] \Rightarrow$$

$$\Rightarrow \mathbf{j} = en(b_i + b_e)\left[\mathbf{E}_{con} - \frac{D_e - D_i}{b_i + b_e}\cdot\frac{\nabla n}{n} - \frac{D_e}{(b_i + b_e)}\frac{\nabla kT_e}{kT_e}\right] + en\left[(D_e - D_i)\frac{\nabla n_e}{n_e} + D_e\frac{\nabla kT_e}{kT_e}\right] \Rightarrow$$

$$\Rightarrow \mathbf{j} = \mathbf{j}_{\ddot{\imath}\eth} + \mathbf{j}_{\nabla_n} + \mathbf{j}_{\nabla_{T_e}},$$

where

$$\mathbf{j}_{con} = en(b_i + b_e)\mathbf{E}_{con} = \sigma\mathbf{E}_{con};$$

$$\mathbf{j}_{\nabla n} = en\left[(D_e - D_i) - (b_i + b_e)\frac{D_e - D_i}{b_i + b_e}\right]\frac{\nabla n}{n};$$

$$\mathbf{j}_{\nabla_{T_e}} = en\left[D_e - (b_i + b_e)\frac{D_e}{(b_i + b_e)}\right]\frac{\nabla kT_e}{kT_e}.$$

Evidently, the current densities $\mathbf{j}_{\nabla_n}$ and $\mathbf{j}_{\nabla_{T_e}}$, generated by diffusion of charged particles under the influence of their concentration gradients and electron temperature, are equal to zero.

The following conclusions can be drawn from this.

First, ambipolar diffusion is prevalent in the given discharge. Indeed, using the obtained expressions for the diffusion flows of charged particles, we obtain:

$$\mathbf{u}_{e_{sc}} = \mathbf{u}_{i_{sc}} = -\left[\frac{(b_iD_e + b_eD_i)}{b_i + b_e}\right]\frac{\nabla n}{n} - \frac{b_iD_e}{b_i + b_e} \times \frac{\nabla kT_e}{kT_e} = -D_{a_{\nabla n}}\frac{\nabla n}{n} - D_{a_{\nabla T_e}}\frac{\nabla kT_e}{kT_e},$$

where

$$D_{a_{\nabla n}} = \frac{(b_iD_e + b_eD_i)}{b_i + b_e},\ D_{a_{\nabla T_e}} = \frac{b_iD_e}{b_i + b_e}$$

are the respective coefficients of ambipolar diffusion.

Given that $D_e >> D_i$, $b_e >> b_i$, $T_e >> T_i$, we obtain:

$$D_{a_{\nabla n}} \approx D_{a_{\nabla T_e}} = D_a = b_i \frac{kT_e}{e}.$$

Secondly, the current density in DC plasma, equal to the sum $\mathbf{j} = \mathbf{j}_{con} + \mathbf{j}_{sc,}$ depends in this case only on the conductivity current density:

$$\mathbf{j} = en(\mathbf{u}_i - \mathbf{u}_e) = \mathbf{j}_{con} = en(b_i + b_e)\mathbf{E}_{con} \approx$$
$$\approx enb_e\mathbf{E}_{cjn} = \sigma_e \mathbf{E}_{con}.$$

The current density $\mathbf{j}_{con}$ can be represented as the sum

$$\mathbf{j}_{con} = \mathbf{j}_{con_r r} + \mathbf{j}_{con_z}.$$

Since the longitudinal component of the conductivity current density $\mathbf{j}_{conz}$ is associated with electron drift along the column axis in the longitudinal electric field $\mathbf{E}_{conz} = \mathbf{E}_z(z)$, the detected discharge current I can be written as

$$I = \int_0^{2\pi}\int_0^{R_z} \mathbf{j}_{con_z} r dr d\theta =$$
$$= 2\pi e b_e E_z(z) \int_0^{R_z} n(r,z) r dr = \text{const}(z), \tag{5}$$

where R_z is the radius R at the point with the coordinate z.

Let us now consider the equations of balance of charged particles:

$$\frac{\partial n_e}{\partial t} = -\nabla(n_e \mathbf{u}_e) + \frac{\delta n_e}{\delta t};$$
$$\frac{\partial n_i}{\partial t} = -\nabla(n_i \mathbf{u}_e) + \frac{\delta n_i}{\delta t}.$$

The following condition should be satisfied to maintain quasi-neutral plasma in steady state:

$$\frac{\partial n_{\mathring{a}}}{\partial t} = \frac{\partial n_i}{\partial t}.$$

In our case, the generation terms $\frac{\delta n_i}{\delta t}$ and $\frac{\delta n_e}{\delta t}$ of the equations (absence of bulk recombination and stepwise ionization) are determined only by direct ionization of atoms by electron impact with the ionization frequency

$$\nu_i = n_a \langle \sigma_{0_i} v_e \rangle.$$

Since bulk ionization and recombination on the walls of the tube lead to simultaneous production or decay of an ion-electron pair, which is expressed as

$$\frac{\delta n_e}{\delta t} = \frac{\delta n_i}{\delta t} = \frac{\delta n}{\delta t} = n\nu_i,$$

we obtain the following equalities: $\nabla n\mathbf{u}_e = \nabla n\mathbf{u}_i = n\nu_i$.

Then, assuming that plasma is quasi-neutral and that the mobilities of electrons and ions, as well as the temperatures of ions are independent of the coordinates, we use expressions (2), (3) for directed velocities of electrons and ions, making simple transformations, excluding $\mathbf{E}$ and provided that $\nabla n\mathbf{u}_e = \nabla n\mathbf{u}_i = n\nu_i$, and obtain the following equation:

$$n\nu_i(b_i + b_e) + (b_i D_e + b_e D_i)\Delta n +$$
$$+\left(b_i D_e \frac{\nabla T_e}{T_e}\right)\nabla n +$$
$$+b_i\left[\nabla D_e \cdot \frac{\nabla T_e}{T_e} + D_e \nabla\left(\frac{\nabla T_e}{T_e}\right)\right] n = 0.$$

Since n = const(θ), n_a = const(r,θ,z), T_e = const(r,θ), and the atom concentration and the electron temperature are independent of the radial coordinates, which in turn means that the electron diffusion coefficient and the ionization frequency are independent of the coordinate r, this equation can be represented as

$$n\nu_i + \frac{(b_i D_e + b_e D_i)}{(b_i + b_e)}(\Delta_r n + \Delta_z n) +$$
$$+\left(\frac{b_i D_e}{(b_i + b_e)} \frac{\nabla_z T_e}{T_e}\right)\nabla_z n + \frac{b_i}{(b_i + b_e)} \times$$
$$\times\left\{\left[\nabla_z D_e \cdot \frac{\nabla_z T_e}{T_e} + D_e \nabla_z\left(\frac{\nabla_z T_e}{T_e}\right)\right]\right\} n = 0.$$

Then, since $b_e >> b_i$, $T_e >> T_i$ and

$$D_{\grave{a}} = \frac{b_i D_e + b_e D_i}{b_i + b_e} \approx b_i \frac{D_e}{b_e} = b_i \frac{kT_e}{e},$$

we obtain the following equation:

$$n\nu_i(b_i + b_e) + (b_i D_e + b_e D_i)\Delta n +$$
$$+\left(b_i D_e \frac{\nabla T_e}{T_e}\right)\nabla n +$$
$$+b_i\left[\nabla D_e \cdot \frac{\nabla T_e}{T_e} + D_e \nabla\left(\frac{\nabla T_e}{T_e}\right)\right] n = 0.$$

After transforming it, we obtain:

$$\left(\Delta_r n+\Delta_z n\right)+\frac{2}{D_a}\left(\nabla_z D_a\right)\nabla_z n+$$

$$+\frac{1}{D_a}\left[\nabla_z\left(\nabla_z D_a\right)+\nu_i\right]n=0.$$

Let us transform this equation so it takes the following form:

$$\frac{\partial^2 n}{\partial r^2}+\frac{1}{r}\frac{\partial n}{\partial r}+\frac{\partial^2 n}{\partial z^2}+$$

$$+\frac{2}{D_a}\frac{dD_a}{dz}\frac{dn}{dz}+\frac{1}{D_a}\left(\frac{d^2 D_a}{dz^2}+\nu_i\right)n=0. \quad (6)$$

Now we introduce the dimensionless concentration

$$N(r,z)=\frac{n(r,z)}{n_0}=f_{n_r}(r)\cdot f_{n_z}(z),$$

where $n_0 = n(0,0)$ is the concentration of charged particles on the axis of the discharge ($r = 0$) at the point $z = 0$; $f_{nr}(r) = \text{const}(z)$ is the function of radius only, $f_{nz}(z) = \text{const}(r)$ is the function of the longitudinal coordinate only, with boundary conditions:

$$f_{n_r}(r=0)=f_{n_z}(z=0)=f_{R_z}(z=0)=1;$$

$$f_{n_r}(R_z)=0; f_{n_z}(l)=\frac{n(0,l)}{n_0}; f_{R_z}=\frac{R_z}{R_0}.$$

$$0\le r\le R_z;\ \ 0\le z\le l;$$

The concentration $n(0,z)$ on the axis ($r = 0$) at the point z is equal to:

$$n(0,z)=n_{0z}(z)=n_0 f_{n_z}(z)=\text{const}(r),$$

and the expression for $n(r,z)$ is written as

$$n(r,z)=n_{0_z}(z)\cdot f_{n_r}(r)=n_0 f_{n_r}(r) f_{n_z}(z).$$

Eq. (6) is converted to the following form:

$$f_{n_z}(z)\frac{d^2 f_{n_r}(r)}{dr^2}+f_{n_z}(z)\frac{1}{r}\frac{df_{n_r}(r)}{dr}+$$

$$+f_{n_r}(r)\frac{d^2 f_{n_z}(z)}{dz^2}+2\frac{f_{n_r}(r)}{D_a}\frac{dD_a}{dz}\frac{df_{n_z}(z)}{dz}+$$

$$+\frac{1}{D_a}\left\{\left[\frac{d^2 D_a}{dz^2}\right]+\nu_i\right\}f_{n_z}(z)f_{n_r}(r)=0,$$

then,

$$\frac{1}{f_{n_r}(r)}\frac{d^2 f_{n_r}(r)}{dr^2}+\frac{1}{f_{n_r}(r)}\cdot\frac{1}{r}\frac{df_{n_r}(r)}{dr}=$$

$$=-\frac{1}{f_{n_z}(z)}\cdot\frac{d^2 f_{n_z}(z)}{dz^2}-\frac{2}{f_{n_z}(z)}\times$$

$$\times\frac{1}{D_a}\frac{dD_a}{dz}\cdot\frac{df_{n_z}(z)}{dz}-\frac{1}{D_a}\left[\frac{d^2 D_a}{dz^2}+\nu_i\right]=-\lambda,$$

or, in a different notation, we obtain a system of equations taking the form

$$\begin{cases}\dfrac{d^2 f_{n_r}(r)}{dr^2}+\dfrac{1}{r}\dfrac{df_{n_r}(r)}{dr}+\lambda f_{n_r}(r)=0\\ \dfrac{d^2 f_{n_z}(z)}{dz^2}+\dfrac{2}{D_a}\dfrac{dD_a}{dz}\cdot\dfrac{df_{n_z}(z)}{dz}+\\ +\dfrac{1}{D_a}\left[\dfrac{d^2 D_a}{dz^2}+\nu_i\right]f_{n_z}(z)-\\ -f_{n_z}(z)\lambda=0\end{cases}\Rightarrow$$

$$\Rightarrow\begin{cases}r^2\dfrac{d^2 f_{n_r}(r)}{dr^2}+r\dfrac{df_{n_r}(r)}{dr}+\\ +\left(r\sqrt{\lambda}\right)^2 f_{n_r}(r)=0\\ \dfrac{d^2 f_{n_z}(z)}{dz^2}+\dfrac{2}{D_a}\dfrac{dD_a}{dz}\cdot\dfrac{df_{n_z}(z)}{dz}+\\ +\dfrac{1}{D_a}\left[\dfrac{d^2 D_a}{dz^2}+\nu_i-D_a\lambda\right]f_{n_z}(z)=0\end{cases}$$

If $\lambda = \text{const}(r)$, the first equation is the zero-order Bessel equation. Its solution is known:

$$f_{n_r}(r)=J_0\left(r\sqrt{\lambda}\right).$$

Then, due to the boundary condition $f_{nr}(R_z) = 0$, we obtain $J_0(R_z\sqrt{\lambda}) = 0$, if

$$\lambda=\left(\frac{2.405}{R_z}\right)^2.$$

Given the expression obtained for λ, the second equation takes the following form after transformations:

$$\frac{d}{dz}\left(D_a\frac{df_{n_z}(z)}{dz}\right)+\frac{d}{dz}\left(f_{n_z}(z)\frac{dD_a}{dz}\right)+$$

$$+\left[\nu_i-D_a\left(\frac{2.405}{R_z}\right)^2\right]f_{n_z}(z)=0. \quad (7)$$

Since the main mechanism for decay of charged particles in discharge is diffusive escape to the tube wall, it is natural to assume that the discharge is stable only at a certain ratio of the ionization frequency ν_i, determining the production time τ_i of charged particles per unit of discharge length, $\tau_i = 1/\nu_i$, and the ambipolar diffusion coefficient D_a, determining the time for diffusive escape of particles from the discharge to the wall of the tube of radius R_z:

$$\tau_{D_a} = R_z^2 / D_a.$$

We assume that the concentration of particles on the wall is equal to zero, and since the distribution of electron concentration over the radius is a zero-order Bessel function whose first root is reached with the argument equal to 2.455, we can take

$$R_z\sqrt{\frac{\nu_i}{D_a}} = 2.405 \text{ or } \left(\frac{2.405}{R_z}\right)^2 - \frac{\nu_i}{D_a} = 0. \quad (8)$$

In other words, we obtained a ‘classical’ relationship between the ionization frequency and the ambipolar diffusion coefficient for the Schottky’s positive column of a direct-current discharge in a cylindrical discharge channel [7, 10] with the radius R_0 replaced by $R_z = R_0 f_R(z)$.

Eq. (7) can then be written as

$$\frac{d}{dz}\left(D_a\frac{df_{n_z}(z)}{dz}\right) + \frac{d}{dz}\left(f_{n_z}(z)\frac{dD_a}{dz}\right) = 0.$$

It can be converted to the form:

$$\frac{d}{dz}\left[D_a f_{n_z}(z)\right] = C \Rightarrow D_a(z) f_{n_z}(z) =$$

$$= Cz \Rightarrow D_a(z) f_{n_z}(z) = Cz + G.$$

Given that $f_{nz}(z = 0) = 1$, we obtain the following expression:

$$D_a(z) f_{n_z} = Cz + D_a(0) \Rightarrow$$

$$\Rightarrow f_{n_z}(z) = \frac{Cz + D_a(0)}{D_a(z)} = \frac{C_1 z + T_e(0)}{T_e(z)}.$$

As established above (see Eq. (5)), the current in the column, equal to the difference in the electron and ion fluxes, depends only on the drift fluxes of charged particles in an external electric field, i.e., only on the conductivity current:

$$I = 2\pi e b_e E_z(z)\int_0^{R_z} n(r,z)\,r dr =$$

$$= \mathrm{const}(z) \Rightarrow$$

$$\Rightarrow I_d = 2\pi e b_e E_z(z) n_0 f_{n_z}(z) \times$$

$$\times \int_0^{R_z} f_{n_r}(r)\,r dr = \mathrm{const}(z).$$

Then, since

$$f_{n_r}(r) = J_0\left(r\sqrt{\lambda}\right),$$

$$\lambda = \left(\frac{2.405}{R_z}\right)^2 = \left(\frac{2.405}{R_0 f_R(z)}\right)^2,$$

using the boundary conditions for R_z and E_z, we obtain:

$$I = 1.36 R_z^2 e b_e n_0 f_{n_z}(z) E_z(z) \Rightarrow$$

$$\Rightarrow n_0 f_{n_z}(z) =$$

$$= \frac{0.737 I_\partial}{R_0^2 f_R^2(z) e b_e E_{0z} f_{E_z}(z)} \Rightarrow$$

$$\Rightarrow n_0 = \frac{0.737 I_d}{R_0^2 e b_e E_{0z}} \Rightarrow$$

$$\Rightarrow f_{n_z}(z) = \frac{1}{f_R^2(z) f_{E_z}(z)}.$$

The average electron concentration over the discharge cross-section is expressed as

$$\overline{n}_z(z) = \frac{\int_0^{R(z)} r n_e dr}{\int_0^{R(z)} r dr} = 0.43 n_0 f_{n_z}(z) =$$

$$= \frac{0.317 I_\partial}{e b_e R_0^2 f_R^2(z) E_{0z} f_{E_z}(z)}.$$

Let us return to expression (8). We obtained that the ionization frequency of atoms $\nu_i(z)$ for the case of ‘diffusion’ discharge in a tube with variable radius is written as follows:

$$\nu_i = D_a\left(\frac{2.405}{R_z}\right)^2 \Rightarrow \nu_i = b_i\frac{kT_e}{e}\left(\frac{2.405}{R_z}\right)^2.$$

Since we assumed from the very beginning that only ‘direct’ ionization occurs in the discharge, the ionization frequency ν_i of atoms (their concentration is denoted as n_a) by electron impact from the ground state is written as

$$\nu_i = n_a \left\langle \sigma_{0_i} v_e \right\rangle,$$

where $\langle \sigma_{0_i} v_e \rangle$ is the ionization reaction rate constant, i.e., the product $\sigma_{0_i} v_e$, averaged over the electron energy distribution function.

In case of Maxwellian electron energy distribution, the direct ionization cross-section $\sigma_{0_i}(\varepsilon_e)$ as function of electron energy ε_e should be approximated by such a straight line

$$\sigma_{0_i} = C_i (\varepsilon_e - \varepsilon_i),$$

which, with $\varepsilon_e \geq \varepsilon_i$, is characterized by the constant C_i.

With this approximation, we obtain the following expression for the ionization frequency:

$$\nu_i(z) = C_i n_a \left[\varepsilon_i + 2kT_e(z)\right] \times \\ \times \sqrt{\frac{8kT_e(z)}{\pi m_e}} \cdot \exp\left(-\frac{\varepsilon_i}{kT_e(z)}\right),$$

then,

$$\nu_i = C_i n_a \sqrt{\frac{8kT_e(z)}{\pi m_e}} \cdot \left(\varepsilon_i + 2kT(z)_e\right) \times \\ \times \exp\left(-\frac{\varepsilon_i}{kT_e(z)}\right) = b_i \frac{kT_e(z)}{e} \left(\frac{2{,}405}{R_z(z)}\right)^2.$$

As a result, we obtain the well-known formula [7, 10] relating kT_e in a PC of a diffusion cylindrical discharge to the concentration of ionizable atoms and the radius of the discharge channel replacing R with $R_z(z) = R_0 f_R(z)$:

$$\frac{\sqrt{\dfrac{\varepsilon_i}{kT_e(z)}} \cdot \exp\left(\dfrac{\varepsilon_i}{kT_e(z)}\right)}{\left(1 + \dfrac{\varepsilon_i}{2kT_e(z)}\right)} = \\ = 0.552 \frac{e}{\sqrt{m_e}} \left(\frac{C_i \sqrt{\varepsilon_i}}{b_i n_a}\right) n_a^2 \times \\ \times \left[R_0 f_R(z)\right]^2 (z).$$

Now let us consider the definition of the field $E_z(z)$. For this purpose, we use the equation of energy balance per unit of column length, following prominent Russian physicists in their classic studies [7, 12]:

$$IE_z = P_v + P_w,$$

where IE_z is the power expended by the longitudinal electric field generated by an external source to accelerate (‘heat’) the electrons in the column. As noted above, P_v is the power associated with the energy acquired by electrons that they spend in elastic collisions with atoms (gas heating); P_w is the power that ions transfer to the wall.

The expressions for P_v and P_w can be written as [7, 12]:

$$P_v = \frac{3}{2} \pi R^2(z) \cdot \overline{n}(z) \chi_{ea} \nu_{ea}(z) kT_e(z);$$

$$P_w = 2\pi R(z) j_{i_w} \cdot \left(U_i + 1.7 \frac{kT_e(z)}{e} + U_w\right),$$

where χ_{ea} is the energy transfer coefficient in elastic electron-atom collisions, $\chi_{ea} = 2m_e/m_a$; j_{iw} is the ion current to the tube wall; U_i is the atomic ionization potential; U_w is the near-wall jump in potential.

Then the power lost by electrons in elastic collisions is expressed as

$$P_v(z) = 4.05 \frac{m_e}{m_a} n_0(z) \cdot kT_e(z) \times \\ \times R_z^2(z) \cdot \nu_{ea} = 3 \frac{I_\delta}{m_a b_e^2 E_z(z)} \cdot kT_e(z).$$

The expression for the near-wall potential jump U_w is found from the condition that electron fluxes Γ_{eg} and ion fluxes Γ_{ig} are equal at the interface between the plasma and the tube wall [7, 8].

Assuming that the coefficient of reflection of electrons and ions from the tube wall is negligible, the directional velocity of ions in the layer is determined by the ambipolar field, and the velocity of electrons by their random velocity, we obtain:

$$\Gamma_{eg} = \frac{1}{\sqrt{2\pi}} n_{eg} \sqrt{\frac{kT_e(z)}{m_e}} \times \\ \times \exp\left(-\frac{eU_w(z)}{kT_e(z)}\right) = \Gamma_{ig} = n_{ig} \sqrt{\frac{kT_e(z)}{m_i}}$$

Here n_{eg}, n_{ig} are the concentrations of electrons and ions at the layer interface. It follows from this equation that

$$U_w(z) = \frac{kT_e(z)}{e} \ln 0.4 \sqrt{\frac{m_i}{m_e}}.$$

The wall current of ions can be written in accordance with the expressions given in [7, 8]:

$$j_{iw}(z) = -en_{(r\approx R_z)} D_a \frac{1}{n_{(r\approx R_z)}} \left(\frac{\partial n}{\partial r}\right)_{(r\approx R_z)} =$$

$$= -eD_a n_0(z) \left(\frac{\partial}{\partial r} f_{n_{e_r}}\right)_{(r\approx R_z)} =$$

$$= -eD_a n_0(z) \left[\frac{\partial}{\partial r} J_0\left(\frac{2.405}{R_z} r\right)\right]_{(r\approx R_z)} =$$

$$= eD_a n_0(z) \frac{2.405}{R(z)} J_1(2.405) \Rightarrow$$

$$\Rightarrow j_{w_i}(z) \approx 3\bar{n}(z) b_i \frac{kT_e(z)}{R_0 f_R(z)}.$$

The power that ions transfer to the wall is then written as

$$P_w(z) == 6\pi R_z \bar{n}(z) \frac{kT_e(z)}{R_0 f_R(z)} b_i \times$$

$$\times \left(U_i + 1.7 \frac{kT_e(z)}{e} + U_w(z)\right),$$

or

$$P_w(z) = \frac{6I}{\left[R_0 f_R(z)\right]^2 E_z(z)} \frac{b_i}{b_e} \cdot \frac{kT_e(z)}{e^2} \times$$

$$\times \left[\mathring{a}U_i + kT_e(z)\left(1{,}7 + \ln 0.4\sqrt{\frac{m_a}{m_e}}\right)\right].$$

The total energy balance then takes the form:

$$IE_z(z) = P_v(z) + P_w(z) =$$

$$= 3\frac{I}{m_{\mathring{a}} b_e^2 E_z(z)} \cdot kT_e(z) +$$

$$+ \frac{6I}{\left[R_0 f_R(z)\right]^2 E_z(z)} \frac{b_i}{b_e} \cdot \frac{kT_e(z)}{e^2} \times$$

$$\times \left[\mathring{a}U_i + kT_e(z)\left(1.7 + \ln 0.4\sqrt{\frac{m_a}{m_e}}\right)\right].$$

From here we obtain:

$$(eE_z)^2 = 3kT_e(z)\frac{m_e \nu_{ea}}{m_a} \times$$

$$\times \left\{ m_e \nu_{ea} + \frac{4}{\left[R_0 f_R(z)\right]^2 \nu_{ia}} \times \right.$$

$$\left. \times \left[eU_i + kT_e(z)\left(1.7 + \ln 0.4\sqrt{\frac{m_i}{m_e}}\right)\right]\right\}.$$

Let us summarize our main results.

Considering the processes in a positive column of direct current discharge under typical discharge conditions in tubes of variable diameter, we obtained equations for the electron concentration (as function of longitudinal and transverse coordinates), electron temperature, and electric field projection, relating them to radius of the discharge channel depending on the longitudinal coordinate. The system of equations obtained provides a solution to the problem. The study presents relatively simple expressions that relate easily controlled external parameters of the column.

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