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DONKIN'S DIFFERENTIAL OPERATORS FOR HOMOGENEOUS HARMONIC FUNCTIONS

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The work continues the study of the Donkin's operators for homogeneous harmonic functions. Previously, a basic list of such first-order operators for three-dimensional harmonic functions was obtained. The objective of this study is to prove that any linear combinations with constant coefficients made up of the Donkin's basic operators are again Donkin's operators. Since the reversibility property is fundamental for such operators, and since the reversibility of each of the linear differential operators taken separately does not automatically imply the reversibility of their linear combination, this statement is nontrivial and requires strict proof. This proof has been given in this paper.

Keywords: electrostatic field, magnetostatic field, scalar potential, homogeneous function, harmonic function

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ДИФФЕРЕНЦИАЛЬНЫЕ ОПЕРАТОРЫ ДОНКИНА ДЛЯ ОДНОРОДНЫХ ГАРМОНИЧЕСКИХ ФУНКЦИЙ

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Данная работа продолжает изучение операторов Донкина для однородных гармонических функций. Ранее был получен базисный список таких операторов первого порядка для трехмерных гармонических функций. Задача настоящего исследования – доказать, что любые линейные комбинации с постоянными коэффициентами, составленные из базисных операторов Донкина, – тоже операторы Донкина. Ввиду того, что свойство обратимости есть фундаментальный признак таких операторов, и поскольку из обратимости каждого из линейных дифференциальных операторов по отдельности не следует автоматически обратимость их линейной комбинации, указанное утверждение является нетривиальным и требует строгого доказательства. Оно представлено в данной статье.

Ключевые слова: электростатическое поле, магнитостатическое поле, скалярный потенциал, однородная функция, гармоническая функция

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Problem statement

This study continues the investigations started in [1–3], pursuing the subject of Thomson transforms and Donkin operators for three-dimensional harmonic (that is, satisfying the Laplace equation) functions that are Euler-homogeneous.

An electric or magnetic field is called Euler-homogeneous with a degree of homogeneity equal to k if the electric field strength $\mathbf{E}$ and/or magnetic flux density $\mathbf{B}$ satisfy the identities

$$\forall\lambda>0:\ \mathbf{E}(\lambda x,\lambda y,\lambda z)\equiv\lambda^{k-1}\mathbf{E}(x,y,z),$$

$$\forall\lambda>0:\ \mathbf{B}(\lambda x,\lambda y,\lambda z)\equiv\lambda^{k-1}\mathbf{B}(x,y,z),$$

at every point in space.

As a rule, such fields are characterized by scalar potentials $U(x,y,z)$, which are Euler--homogeneous (or, more precisely, positively homogeneous, i.e., with $\lambda > 0$) in the sense given to this term in classical mathematical analysis [4, 5]:

$$\forall\lambda>0:\ U(\lambda x,\lambda y,\lambda z)\equiv\lambda^{k}U(x,y,z).$$

Homogeneous scalar and/or vector potentials for Euler-homogeneous fields are explored in detail in [6]. Importantly, the Euler differential relation for homogeneous functions is satisfied at each point in space for differentiable functions U that are Euler-homogeneous of degree k [4, 5]:

$$x\frac{\partial U}{\partial x}+y\frac{\partial U}{\partial y}+z\frac{\partial U}{\partial z}-kU=0. \qquad (1)$$

This relation is both necessary and sufficient. This includes the following statements:

a) if the function is homogeneous with a degree of homogeneity k and at the same time differentiable everywhere, then equality (1) is satisfied for it at each point in space;

b) if equality (1) is satisfied for a function that is differentiable everywhere at each point in space, then this function is Euler-homogeneous, with a degree of homogeneity k.

Elegant proof of statement b) can be found, for example, in [4].

Strict definitions of the terms "Thomson transform" and "Donkin operator" have been already given in [3] (also published in this issue) and are therefore not repeated here. However, we should stress that there is a fundamental difference between the Thomson transforms and the Donkin operators: the Thomson transforms preserve the function obtained harmonic and homogeneous, but, unlike Donkin operators, they do not guarantee that the transformation is invertible. Donkin operators, on the other hand, are invertible in the sense that there is a prototype function for any homogeneous harmonic function (this prototype is also homogeneous and harmonic), from which the given homogeneous harmonic function is obtained using this operator.

The term "Thomson transform" is used to avoid confusion with the linear differential operators of the general form considered here with the original algebraic Thomson formula (Kelvin transform) [9–16]. The Thomson formula transforms three-dimensional harmonic functions into new three-dimensional harmonic functions in accordance with the rule

$$V(x,y,z)=\frac{1}{r}U\left(\frac{x}{r^2},\frac{y}{r^2},\frac{z}{r^2}\right), \qquad (2)$$

where $r=\sqrt{x^2+y^2+z^2}$ (from now on).

Eq. (2) not only transforms three-dimensional harmonic functions U *of a* general form into new three-dimensional harmonic functions V (this can be verified by direct substitution) but also transforms homogeneous functions U *of* degree k to new homogeneous functions V of degree $(-k-1)$. The common factor $1/r^2$ for homogeneous functions can be taken out of the function sign in Eq. (2), so that it can be written in simplified form as

$$V(x,y,z)=r^{-2k-1}U(x,y,z). \qquad (3)$$

It is clear that the mathematical expression (3) is homogeneous, and it follows that it is harmonic for homogeneous harmonic functions U from the equality (see [17, 18 (Appendix B to Chapter 1)]):

$$\begin{aligned}V_{xx}+V_{yy}+V_{zz}&\equiv r^m\left(U_{xx}+U_{yy}+U_{zz}\right)+\\&+2mr^{m-2}\left(xU_x+yU_y+zU_z-kU\right)+\\&+m(m+2k+1)r^{m-2}U,\end{aligned} \qquad (4)$$

which is satisfied for any functions of the form

$$V(x,y,z) = r^m U(x,y,z)$$

with an arbitrary exponent m, for an arbitrary parameter k and an arbitrary function U.

However, transformation (3) no longer converts harmonic functions of a general form into new harmonic functions; expression (3) is preserved harmonic only for those harmonic functions that are Euler-homogeneous with a degree of homogeneity k (i.e., which satisfy Euler differential relation (1)).

The results published in [1–3] indicate

that the complete list of elementary Thomson transforms for homogeneous harmonic functions of degree k (the Thomson transform is a linear differential operator of the first order) includes the following expressions:

$$V(x,y,z)=U(x,y,z), \tag{5}$$

$$V(x,y,z)=U_x(x,y,z), \tag{6}$$

$$V(x,y,z)=U_y(x,y,z), \tag{7}$$

$$V(x,y,z)=U_z(x,y,z), \tag{8}$$

$$V(x,y,z)= =(2k+1)xU(x,y,z)-r^2U_x(x,y,z), \tag{9}$$

$$V(x,y,z)= =(2k+1)yU(x,y,z)-r^2U_y(x,y,z), \tag{10}$$

$$V(x,y,z)= =(2k+1)zU(x,y,z)-r^2U_z(x,y,z), \tag{11}$$

$$V(x,y,z)=yU_x(x,y,z)-xU_y(x,y,z), \tag{12}$$

$$V(x,y,z)=zU_x(x,y,z)-xU_z(x,y,z), \tag{13}$$

$$V(x,y,z)=zU_y(x,y,z)-yU_z(x,y,z), \tag{14}$$

$$V(x,y,z)=\frac{1}{r^{2k+1}}U(x,y,z), \tag{15}$$

$$V(x,y,z)=\frac{1}{r^{2k-1}}U_x(x,y,z), \tag{16}$$

$$V(x,y,z)=\frac{1}{r^{2k-1}}U_y(x,y,z), \tag{17}$$

$$V(x,y,z)=\frac{1}{r^{2k-1}}U_z(x,y,z), \tag{18}$$

$$V(x,y,z)=\frac{(2k+1)x}{r^{2k+3}}U(x,y,z)- -\frac{1}{r^{2k+1}}U_x(x,y,z), \tag{19}$$

$$V(x,y,z)=\frac{(2k+1)y}{r^{2k+3}}U(x,y,z)- -\frac{1}{r^{2k+1}}U_y(x,y,z), \tag{20}$$

$$V(x,y,z)=\frac{(2k+1)z}{r^{2k+3}}U(x,y,z)- -\frac{1}{r^{2k+1}}U_z(x,y,z), \tag{21}$$

$$V(x,y,z)=\frac{y}{r^{2k+1}}U_x(x,y,z)- -\frac{x}{r^{2k+1}}U_y(x,y,z), \tag{22}$$

$$V(x,y,z)=\frac{z}{r^{2k+1}}U_x(x,y,z)- -\frac{x}{r^{2k+1}}U_z(x,y,z) \tag{23}$$

$$V(x,y,z)=\frac{z}{r^{2k+1}}U_y(x,y,z)- -\frac{y}{r^{2k+1}}U_z(x,y,z) \tag{24}$$

where the subscripts x, y, and z denote the partial derivatives with respect to the corresponding variables.

It was established in [3] that each of the elementary transformations (5)–(24), considered separately, is invertible on the set of homogeneous harmonic functions, i.e., it is a basic Donkin operator.

Evidently, any linear combination with constant coefficients, composed of basic Eqs. (5)–(24) corresponding to the same degree of homogeneity, transforms homogeneous harmonic functions into new homogeneous harmonic functions and, therefore, belongs to the class of Thomson transforms. However, the invertibility of such transformations on a subset of homogeneous harmonic functions with a given degree of homogeneity is not obvious and should be further investigated.

The goal of this study is to establish whether linear superpositions with constant coefficients composed of elementary Donkin operators (5)–(24) are composite Donkin operators in the sense of the definitions given earlier.

Relationship between three-dimensional homogeneous harmonic functions and two-dimensional elliptic equations

Now we are going to make a transition from three-dimensional homogeneous harmonic functions to two-dimensional functions that satisfy some auxiliary two-dimensional elliptic equations. This technique is of particular scientific interest, so let us consider it in more detail.

There are Donkin formulas for three-dimensional harmonic functions of degree zero and degree −1 [7, 8, 19–24]:

$$V_0(x,y,z) = H\left(\frac{x}{z+r}, \frac{y}{z+r}\right), \tag{25}$$

$$V_{-1}(x,y,z) = \frac{1}{r} H\left(\frac{x}{z+r}, \frac{y}{z+r}\right), \tag{26}$$

which establish a one-to-one correspondence between the solutions $H(p,q)$ of the two-dimensional Laplace equation

$$H_{pp} + H_{qq} = 0$$

and three-dimensional homogeneous harmonic functions with homogeneity degrees 0 and −1. Similarly, by substituting the Donkin coordinates

$$\begin{cases} p = \dfrac{x}{z+\sqrt{x^2+y^2+z^2}}, \\ q = \dfrac{y}{z+\sqrt{x^2+y^2+z^2}} \\ r = \sqrt{x^2+y^2+z^2}, \end{cases} \Leftrightarrow \tag{27}$$

$$\Leftrightarrow \begin{cases} x = \dfrac{2pr}{1+p^2+q^2}, \\ y = \dfrac{2qr}{1+p^2+q^2}, \\ z = \pm\dfrac{r\left(1-p^2-q^2\right)}{1+p^2+q^2}. \end{cases}$$

any homogeneous function $U(x,y,z)$ of degree m can be expressed as

$$U(x,y,z) = r^m F\left(\frac{x}{z+r}, \frac{y}{z+r}\right). \tag{28}$$

This formulation is a slightly modified form of the universal representation

$$f(x_1, x_2, \dots x_n) = x_1^k h(x_2/x_1, \dots x_n/x_1)$$

for homogeneous functions of degree k [4, 5].

The necessary and sufficient condition for function (28) to be harmonic (satisfying the three-dimensional Laplace equation) is that the function $F(p,q)$ satisfy the two-dimensional elliptic partial differential equation:

$$\frac{\partial^2 F(p,q)}{\partial p^2} + \frac{\partial^2 F(p,q)}{\partial q^2} + \frac{4m(m+1)}{\left(1+p^2+q^2\right)^2} F(p,q) = 0. \tag{29}$$

Substitution (28), (34) is one-to-one, which is to say that:

a) if $U(x,y,z)$ is a homogeneous harmonic function of degree m, then there exists a function $F(p,q)$ that can be used to represent the function U in form (28), while the function F should obey Eq. (34);

b) if the function $F(p,q)$ obeys Eq. (34), then the function $U(x,y,z)$, calculated by rule (28), is a homogeneous harmonic function of degree m;

c) different functions $U(x,y,z)$ correspond to different functions $F(p,q)$ and vice versa. For example, if we analyze the differential relation between a homogeneous harmonic function $U(x,y,z)$ of degree m and a homogeneous harmonic function $V(x,y,z)$ of degree k, the substitution

$$U(x,y,z) = r^m F\left(\frac{x}{z+r}, \frac{y}{z+r}\right), \tag{30}$$

$$V(x,y,z) = r^k G\left(\frac{x}{z+r}, \frac{y}{z+r}\right) \tag{31}$$

allows, without loss of generality, to reduce the problem to analysis of the differential relation between the functions of two variables $F(p,q)$ and $G(p,q)$, which obey the equations

$$\frac{\partial^2 F(p,q)}{\partial p^2} + \frac{\partial^2 F(p,q)}{\partial q^2} + \frac{4m(m+1)}{\left(1+p^2+q^2\right)^2} F(p,q) = 0, \tag{32}$$

$$\frac{\partial^2 G(p,q)}{\partial p^2} + \frac{\partial^2 G(p,q)}{\partial q^2} + \frac{4k(k+1)}{\left(1+p^2+q^2\right)^2} G(p,q) = 0. \tag{33}$$

Substitution (28) is not the only one possible. We can equally use the substitutions

$$\begin{cases} U(x,y,z) = (z+r)^m F\left(\dfrac{x}{z+r}, \dfrac{y}{z+r}\right), \\ \left(1+p^2+q^2\right)\left(\dfrac{\partial^2 F}{\partial p^2} + \dfrac{\partial^2 F}{\partial q^2}\right) - \\ -4mp\dfrac{\partial F}{\partial p} - 4mq\dfrac{\partial F}{\partial q} + 4m^2 F = 0; \end{cases} \tag{34}$$

$$\begin{cases} U(x,y,z) = r^m F\left(\dfrac{x}{r}, \dfrac{y}{r}\right), \\ (1-p^2)\dfrac{\partial^2 F}{\partial p^2} - 2pq\dfrac{\partial^2 F}{\partial p \partial q} + \\ +(1-q^2)\dfrac{\partial^2 F}{\partial q^2} - 2p\dfrac{\partial F}{\partial p} - \\ -2q\dfrac{\partial F}{\partial q} + m(m+1)F = 0; \end{cases} \tag{35}$$

$$\begin{cases} U(x,y,z) = z^m F\left(\dfrac{x}{z}, \dfrac{y}{z}\right), \\ (1+p^2)\dfrac{\partial^2 F}{\partial p^2} + 2pq\dfrac{\partial^2 F}{\partial p \partial q} + \\ +(1+q^2)\dfrac{\partial^2 F}{\partial q^2} - 2(m-1)p\dfrac{\partial F}{\partial p} - \\ -2(m-1)q\dfrac{\partial F}{\partial q} + m(m-1)F = 0 \end{cases} \tag{36}$$

and so on.

The substitution used to solve a specific problem can be chosen depending on how convenient it is to manipulate a two-dimensional elliptic partial differential equation for the given problem, as well as on aesthetic preferences of the researcher.

Complete systems of first-order partial differential equations

The theorem considered in this section is used as auxiliary (lemma) to prove the main theorem on invertibility of linear superpositions of basic Donkin operators with the same degree of homogeneity; it is given in the next section.

Let there be *m* functions $f_1, f_2, \ldots, f_m$ depending on *n* variables $x_1, x_2, \ldots, x_n$. Let us consider a system of *mn* partial differential equations; all possible first-order partial derivatives of the functions $f_1, f_2, \ldots, f_m$ with respect to the variables $x_1, x_2, \ldots, x_n$ are used on the left-hand side of these equations, and continuously differentiable functions Φ_i^k depending on unknown functions $f_1, f_2, \ldots, f_m$ and independent variables $x_1, x_2, \ldots, x_n$ are used on the right-hand side:

$$i = 1, \ldots n, \; k = 1, \ldots m:$$

$$\frac{\partial f_k}{\partial x_i} = \Phi_i^k\left(f_1, \ldots f_m, x_1, \ldots x_n\right). \tag{37}$$

Let us assume that both the functions $f_1, f_2, \ldots, f_m$ and the functions $\Phi_1^1, \Phi_2^1, \ldots, \Phi_n^m$ are continuously differentiable as many times as necessary to safely carry out subsequent differentiation and for the corresponding theorems on existence and uniqueness of a solution to a system of equations to hold true. Mixed partial derivatives of continuously differentiable functions $f_1, f_2, \ldots, f_m$ with respect to the variables $x_1, x_2, \ldots, x_n$ are independent of the order of differentiation:

$$i = 1, \ldots n, \; k = 1, \ldots m:$$

$$\frac{\partial}{\partial x_i}\left(\frac{\partial f_k}{\partial x_j}\right) \equiv \frac{\partial}{\partial x_j}\left(\frac{\partial f_k}{\partial x_i}\right). \tag{38}$$

The conditions that the right-hand sides of Eqs. (37) should satisfy and that should be fulfilled if system of Eqs. (37) has a non-empty set of solutions follow from identity (38):

$$\begin{aligned} L_{ij}^k = &\left(\frac{\partial \Phi_j^k}{\partial f_1}\Phi_i^1 + \cdots + \frac{\partial \Phi_j^k}{\partial f_m}\Phi_i^m + \frac{\partial \Phi_j^k}{\partial x_i}\right) - \\ &-\left(\frac{\partial \Phi_i^k}{\partial f_1}\Phi_j^1 + \cdots + \frac{\partial \Phi_i^k}{\partial f_m}\Phi_j^m + \frac{\partial \Phi_i^k}{\partial x_j}\right) = 0. \end{aligned} \tag{39}$$

The following cases are possible:

a) all conditions expressed by Eq. (39) are identically equal to zero;

b) there is one or more algebraic equations with respect to unknown $f_1, f_2, \ldots, f_m$ among conditions (39), which are not identically equal to zero and are compatible with each other, so some of the functions $f_1, f_2, \ldots, f_m$ can be algebraically expressed in terms of the remaining functions and independent variables $x_1, x_2, \ldots, x_n$;

c) there is one or more algebraic equations among conditions (39) with respect to unknown functions $f_1, f_2, \ldots, f_m$, which are not identically equal to zero and are compatible with each other, so that *all* functions $f_1, f_2, \ldots, f_m$ can be algebraically expressed in terms of the independent variables $x_1, x_2, \ldots, x_n$;

d) there are algebraic equations among conditions (39) with respect to unknown functions $f_1, f_2, \ldots, f_m$, which are not compatible with each other;

e) there is an algebraic equation among conditions (39), which is not identically equal to zero and does not contain unknown functions $f_1, f_2, \ldots, f_m$, establishing an algebraic relationship between the independent variables $x_1, x_2, \ldots, x_n$.

Conditions d) and e) can actually be

combined into one class of cases. Conditions (39) are incompatible with each other, which means that by using a subset of conditions (39) that are compatible with each other, some or all of the functions $f_1, f_2,..., f_m$ can be algebraically expressed in terms of the remaining functions and independent variables x_1, x_2, ..., x_n and substituted into the remaining equations, with at least one algebraic equation which is not identically equal to zero formed, establishing an algebraic relationship between the independent variables x_1, x_2, ..., x_n. Similarly, an algebraic equation which does not contain unknown functions $f_1, f_2,..., f_m$ and establishes an algebraic relationship between the independent variables x_1, x_2, ..., x_n can be regarded as an algebraic equation for unknown functions f_1, f_2,..., f_m incompatible with other algebraic equations.

Obviously, system of Eqs. (37) cannot have solutions in cases d) or e).

In case c), unknown functions f_1, f_2,..., f_m can be found from the obtained algebraic equations; substituting them in system (37), we can verify either that the functions $f_1, f_2,..., f_m$ found actually satisfy Eqs. (37), or that system of Eqs. (37) has no solutions.

The situation is the same in case b). After some of the functions f_1, f_2,..., f_m are algebraically expressed in terms of the remaining functions and independent variables x_1, x_2, ..., x_n, they can be substituted into Eqs. (37); then we obtain either a complete system of differential equations of the form (37) with respect to a smaller number of unknown functions, or, in addition to differential equations, new algebraic relations for unknown functions. The process of algebraically excluding the functions $f_1, f_2,..., f_m$ from system of Eqs. (37) can then be continued. In particular, if a non-zero algebraic expression that does not contain unknown functions $f_1, f_2,..., f_m$ but only independent variables x_1, x_2, ..., x_n is obtained, this means that system of Eqs. (37) has no solutions.

Finally, in case a), when all conditions (39) are identically equal to zero, the system of identically satisfied equalities (39) turns out to be not only a necessary but also a sufficient condition for system of Eqs. (37) to have solutions.

Lemma 1. *If all relations (39) are identically equal to zero, then the complete system of Eqs. (37) has solutions, and these solutions can be found up to m constants c_1, c_2,..., c_m, chosen arbitrarily.*

Proof of this lemma can be found in [25] (see Chapter IV).

The case when the system of partial differential equations is not complete (the equations are not given for some partial derivatives of unknown functions) is much more complicated for analysis. The theory can be found in [26—29].

Invertibility of linear combinations of basic Donkin operators

Let us consider invertible linear combinations with constant coefficients composed of Eqs. (5)—(14) and (15)—(24) with the same degree of homogeneity for the resulting function. Groups of elementary transformations from the list (5)—(24), given below, transform homogeneous harmonic functions of degree *m* into homogeneous harmonic functions of the same degree:

1) Eqs. (5), (12)—(14) transform one homogeneous function of degree *m* into a new homogeneous function of degree *m*;

2) Eqs. (6)—(8) transform a homogeneous function of degree *m* into a new homogeneous function of degree $m - 1$;

3) Eqs. (9)—(11) transform a homogeneous function of degree *m* into a new homogeneous function of degree $m + 1$;

4) Eqs. (15), (22)—(24) transform a homogeneous degree function *m* into a new homogeneous function of degree $-m - 1$;

5) Eqs. (16)—(18) transform a homogeneous function of degree *m* into a new homogeneous function of degree $-m$;

6) Eqs. (19)—(21) transform a homogeneous function of degree *m* into a new homogeneous function of degree $-m - 2$.

These rules determine which formulas from (5)—(24) can be combined in one linear combination with constant coefficients, so that a homogeneous harmonic function is obtained as a result.

Theorem. *Linear combinations with constant coefficients, composed of either basic Donkin operators (5), (12)—(14), or basic Donkin operators (6)—(8), or basic Donkin operators (9)—(11), or basic Donkin basis operators (15), (22)—(24), or basic Donkin operators (16)—(18), or basic Donkin operators (19)—(21) are invertible on subsets of homogeneous harmonic functions of the corresponding degrees.*

The proof consists of a series of separate independent proofs for each group of basic Donkin operators.

Lemma 2. *Linear combinations with constant coefficients, composed of basic Donkin operators (6)—(8) for the degree of homogeneity $m - 1$, are invertible on a subset of homogeneous harmonic functions of degree $m - 1$.*

Proof. *A linear combination with constant coefficients of Eqs. (6)–(8) has the form*

$$L[U] = aU_x + bU_y + cU_z$$

and corresponds to differentiating the function U with respect to a fixed direction (a,b,c). In accordance with the definition given earlier, this linear combination is a Donkin operator for coefficients that are not simultaneously equal to zero.

To prove Lemma 2, it is sufficient to apply the rotation of the coordinate system relative to the origin, so that the nonzero vector (a,b,c) coincides with one of the coordinate axes, and to use the theorem on differentiation of homogeneous harmonic functions [23, 30]:

a) the rotation relative to the origin preserves the given function V both homogeneous and harmonic;

b) the differentiation operator along one of the new coordinate axes is the basic Donkin operator; therefore, the transformed homogeneous and harmonic function V has a homogeneous and harmonic prototype U;

c) the reverse rotation relative to the origin returns the coordinate system and the function V to the previous state, and also preserves the transformed function U homogeneous and harmonic, at the same time establishing a relationship between these functions, i.e.,

$$V(x,y,z) = aU_x(x,y,z) + \\ + bU_y(x,y,z) + cU_z(x,y,z)$$

Lemma 2 is proved.

Lemma 3. *Linear combinations with constant coefficients composed of basic Donkin operators (9)–(11) for the degree of homogeneity m + 1, or from basic Donkin operators (16)–(18)for the degree of homogeneity –m, or from basic Donkin operators (19)–(21) for the degree of homogeneity –m – 2 are invertible on subsets of homogeneous harmonic functions of the corresponding degrees.*

Proof. This statement follows from the close relationship established by the Thomson formula (3) between these operators and differentiation operators (6)–(8), for which, as we have just established, linear combinations with constant coefficients are invertible transformations of homogeneous harmonic functions.

Indeed, the invertibility of linear combinations composed of Eqs. (16)–(18) follows from the invertibility of linear combinations composed of Eqs. (6)–(8) and the fact that the Thomson formula (3) is invertible (repeatedly applying Eqs. (2), (3) returns the function to its original form). Similarly, linear combinations composed of Eqs. (19)–(21) are invertible if and only if linear combinations composed of Eqs. (9)–(11) invertible. The latter are actually the product of successively applying transformation (3), transformations (6)–(8) and once again transformation (3). For example, the proof that linear combinations composed of Eqs. (9)–(11) are invertible is as follows:

a) a homogeneous harmonic function $V(x,y,z)$ of degree $m + 1$ has a homogeneous and harmonic prototype $V^*(x,y,z)$ of degree $-m - 2$, such that this function is calculated from this prototype in accordance with the Thomson formula (3), in the form

$$V(x,y,z) = V^*(x,y,z)\,r^{2m+3};$$

b) a homogeneous harmonic function $V^*(x,y,z)$ of degree $-m - 2$ has a homogeneous and harmonic prototype $U^*(x,y,z)$ of degree $-m - 1$, such that this function can be obtained from this prototype by differentiating with respect to a fixed direction (a,b,c):

$$V^*(x,y,z) = -aU_x^*(x,y,z) - \\ -bU_y^*(x,y,z) - cU_z^*(x,y,z),$$

where not all coefficients a, b, c are equal to zero;

c) a homogeneous harmonic function $U^*(x,y,z)$ of degree $-m - 1$ has a homogeneous and harmonic prototype $U(x,y,z)$ of degree m, such that this function can be calculated from this prototype in accordance with the Thomson formula (3), in the form

$$U^*(x,y,z) = U(x,y,z)/r^{2m+1}.$$

In this case, the homogeneous harmonic function $U(x,y,z)$ of degree m turns out to be a prototype function; applying the transformation

$$V(x,y,z) = (2m+1)(ax+by+cz)\times \\ \times U(x,y,z) - r^2\big(aU_x(x,y,z) + \\ + bU_y(x,y,z) + cU_z(x,y,z)\big) \quad (40)$$

to this prototype, we obtain the initial homogeneous harmonic function $V(x,y,z)$ of degree $m + 1$.

Notably, transformation (40) is a superposition of transformations c) + b) + a), which can be easily verified by direct calculation of the given superposition.

Therefore, linear combination (40),

composed of basic Donkin operators (9)–(11), is again a Donkin operator, if not all coefficients a, b, c are equal to zero.

Lemma 3 is proved.

Lemma 4. *Linear combinations with constant coefficients composed of basic Donkin operators (5), (12)–(14) for the degree of homogeneity m, or from basic Donkin operators (15), (22)–(24) for the degree of homogeneity –m – 1 are invertible on subsets of homogeneous harmonic functions of the corresponding degrees.*

Proof. Because Thomson formula (3) is invertible, linear combinations composed of Eqs. (15), (22)–(24) are invertible if and only if linear combinations composed of Eqs. (5), (12)–(14) are invertible.

Let us consider an invertible linear combination with constant coefficients, composed of Eqs. (5), (12)–(14):

$$V(x,y,z) = a\left(yU_z - zU_y\right) + b\left(zU_x - xU_z\right) + c\left(xU_y - yU_x\right) + eU, \tag{41}$$

where a, b, c, e are constants, not all of which are equal to zero.

Without loss of generality, we can assume that either $a \neq 0$, or $b \neq 0$, or $c \neq 0$ (if $a = b = c = 0$, then combined operator (41) is transformed into basic operator (5), which is obviously invertible). This requirement imposed so that the expression

$$2apq - b(1 + p^2 - q^2) + 2cq,$$

which is to be divided further, does not vanish identically.

After substituting (30), (31) with $k = m$ into linear combination (41) and into the Laplace equations, we obtain an overdetermined system of partial differential equations for the functions U and V:

$$\frac{\partial^2 F(p,q)}{\partial p^2} + \frac{\partial^2 F(p,q)}{\partial q^2} + \frac{4m(m+1)}{\left(1+p^2+q^2\right)^2} F(p,q) = 0, \tag{42}$$

$$\frac{\partial F(p,q)}{\partial p}\left(b\left(1+p^2-q^2\right) - 2apq - 2cq\right) + \frac{\partial F(p,q)}{\partial q}\left(-a\left(1-p^2+q^2\right) + 2bpq + 2cp\right) + 2eF(p,q) = G(p,q), \tag{43}$$

which should be checked for solutions for an unknown function F, provided that the function G *is* known and satisfies the equation

$$\frac{\partial^2 G(p,q)}{\partial p^2} + \frac{\partial^2 G(p,q)}{\partial q^2} + \frac{4m(m+1)}{\left(1+p^2+q^2\right)^2} G(p,q) = 0. \tag{44}$$

Let us introduce a new unknown function $R(p,q)$ using the substitution

$$\frac{\partial F(p,q)}{\partial q} = R(p,q)F(p,q). \tag{45}$$

We can use Eq. (43), taking into account relation (45), to express the derivative

$$\frac{\partial F(p,q)}{\partial p} = \frac{-G(p,q) + F(p,q)\left(2e + R(p,q)A(p,q)\right)}{B(p,q)}, \tag{46}$$

where

$$A(p,q) = 2bpq - a\left(1-p^2+q^2\right) + 2cp,$$

$$B(p,q) = 2apq - b\left(1+p^2-q^2\right) + 2cq.$$

The condition that mixed derivatives be equal, i.e.,

$$\partial\left(\partial F/\partial p\right)/\partial q = \partial\left(\partial F/\partial q\right)/\partial p,$$

yields an additional linear relation, which the derivatives $\partial R/\partial p$ and $\partial R/\partial q$ should satisfy.

The derivative $\partial^2 F/\partial q^2$ can be found from condition (45) after differentiation with respect to q, and the derivative $\partial^2 F/\partial p^2$ in the form of algebraic expressions containing F, R, $\partial R/\partial p$ and $\partial R/\partial q$ can be found from condition (46), after differentiation with respect to p. Eq. (42) then allows to construct another independent linear relation, which the derivatives $\partial R/\partial p$ and $\partial R/\partial q$ should satisfy.

The derivatives $\partial R/\partial p$ and $\partial R/\partial q$ can be found from these relations as functions of $F(p,q)$, $R(p,q)$, $G(p,q)$, partial derivatives of the first order $\partial G(p,q)/\partial p$ and $\partial G(p,q)/\partial q$, independent variables p, q and constants a, b, c, e.

As a result, we have obtained a complete system of differential equations described in the section "Complete systems of first-order partial differential equations", with unknown functions $F(p,q)$, $R(p,q)$ and independent variables p, q (the function $G(p,q)$ is assumed to be known). We can verify that conditions (39) for the solvability of the resulting complete system of differential equations are identically equal to zero, provided that the function $G(p,q)$ satisfies Eq. (44).

Consequently, the complete system of partial differential equations obtained has a solution (determined up to two arbitrary constants). Therefore, the transformation of homogeneous harmonic functions (41) is invertible on the set of homogeneous harmonic functions, and linear differential operator (41) turns out to be the Donkin operator for any constants a, b, c, e that are not simultaneously equal to zero.

Special case

We considered the degenerate case of the Thomson transform $U(x,y,z) \to V(x,y,z)$ for homogeneous harmonic functions when the functions $U(x,y,z)$ and $V(x,y,z)$ have degrees of homogeneity 0 or −1 [2]. This case is not included in the list of elementary Thomson transforms given above, and should be considered separately. In this section, we discuss the invertibility of the Thomson transform corresponding to this case.

According to Donkin equations (25), (26), the functions $U(x,y,z)$ and $V(x,y,z)$ for the degenerate case of the Thomson transform can be represented as

$$\begin{cases} U(x,y,z) = F\left(\frac{x}{z+r}, \frac{y}{z+r}\right), \\ V(x,y,z) = G\left(\frac{x}{z+r}, \frac{y}{z+r}\right); \end{cases} \tag{47}$$

$$\begin{cases} U(x,y,z) = F\left(\frac{x}{z+r}, \frac{y}{z+r}\right), \\ V(x,y,z) = \frac{1}{r} G\left(\frac{x}{z+r}, \frac{y}{z+r}\right); \end{cases} \tag{48}$$

$$\begin{cases} U(x,y,z) = \frac{1}{r} F\left(\frac{x}{z+r}, \frac{y}{z+r}\right), \\ V(x,y,z) = G\left(\frac{x}{z+r}, \frac{y}{z+r}\right); \end{cases} \tag{49}$$

$$\begin{cases} U(x,y,z) = \frac{1}{r} F\left(\frac{x}{z+r}, \frac{y}{z+r}\right), \\ V(x,y,z) = \frac{1}{r} G\left(\frac{x}{z+r}, \frac{y}{z+r}\right), \end{cases} \tag{50}$$

where the functions $F(p,q)$ and $G(p,q)$ satisfy the two-dimensional Laplace equation.

The relationship between the functions $F(p,q)$ and $G(p,q)$ is established using a linear differential equation of the first order

$$G(p,q) = \lambda F(p,q) + \\ + v(p,q)\frac{\partial F(p,q)}{\partial p} + w(p,q)\frac{\partial F(p,q)}{\partial q}, \tag{51}$$

where λ is an arbitrary real constant, and the functions $v(p,q)$ and $w(p,q)$ satisfy the Cauchy–Riemann equations

$$v_p = w_q,\ v_q = -w_p,$$

that is, they are the real and the imaginary part of some analytic function of a complex variable [31–34] (and each of these functions thus satisfies the two-dimensional Laplace equation).

Linear combinations with constant coefficients composed of expressions of the form (51) with different constants λ_k and different functions $v_k(p,q)$ and $w_k(p,q)$ but with the same harmonic function $F(p,q)$ are obviously also expressions of the form (51) for appropriately chosen constants λ and the functions $v(p,q)$ and $w(p,q)$ satisfying the Cauchy–Riemann equations.

The physical meaning of Eq. (51) is quite clear. If the function $F(p,q)$ is harmonic, it can be regarded the real part of the analytic function of the complex variable $s = p + iq$:

$$f(s) = f(p + iq) = F(p,q) + i\hat{F}(p,q);$$

the functions $F(p,q)$ and $\hat{F}(p,q)$ here are related by the Cauchy–Riemann equations:

$$F_p = \hat{F}_q,\ F_q = -\hat{F}_p.$$

Notably, in accordance with the theorem in the section “Complete systems of first-order partial differential equations”, the Cauchy–Riemann equations, considered as a complete system of partial differential equations with respect to the unknown function $\hat{F}(p,q)$ (for a given function $F(p,q)$) are guaranteed to have a solution up to an arbitrary additive constant when $F(p,q)$ satisfies the Laplace equation.

Returning to discussion of the physical meaning of Eq. (51), we can argue that the analytic function of a complex variable is similarly the function

$$u(s) = u(p + iq) = v(p,q) + iw(p,q),$$

with the functions $v(p, q)$ and $w(p, q)$ related by the Cauchy – Riemann equations.

In this case, the expression

$$\lambda f(s) + u(s)\, df(s)/ds$$

is an analytic function of a complex variable, and its real part, coinciding with the right-hand side of Eq. (51), is a harmonic function (satisfying the two-dimensional Laplace equation).

We can also verify by direct substitution that function (51) is harmonic when the function F is harmonic:

$$\begin{aligned} G_{pp}+G_{qq} &= \lambda\left(F_{pp}+F_{qq}\right)+ \\ &+\left(v_{pp}+v_{qq}\right)F_p+\left(w_{pp}+w_{qq}\right)F_q+ \\ &+v\left(F_{ppp}+F_{pqq}\right)+w\left(F_{ppq}+F_{qqq}\right)+ \\ &+2\left(v_pF_{pp}+w_qF_{qq}\right)+2\left(v_q+w_p\right)F_{pq}=0. \end{aligned} \tag{52}$$

Linear combinations with constant coefficients composed of expressions of the form (51) are also obviously expressions of the form (51) for the appropriately chosen constants λ and the functions $v(p,q)$ and $w(p,q)$ satisfying the Cauchy–Riemann equations.

Since the partial derivatives of the function F with respect to p and q can be expressed using Eqs. (47)–(49) or (50) in terms of the partial derivatives of the function U with respect to x and y, Eq. (51) relating the functions F and G generates a linear differential equation relating the functions U and V, which are therefore the Thomson transform for the given homogeneous functions U and V.

If relation (51) is invertible, the resulting Thomson transform for homogeneous harmonic functions is also invertible, i.e., it is a Donkin operator.

As noted above, when the function $F(p,q)$ satisfies the two-dimensional Laplace equation, it can be regarded as the real part of some analytic function of a complex variable:

$$f(s)=f(p+iq)=F(p,q)+i\hat{F}(p,q), \tag{53}$$

where the real part $F(p,q)$ and the imaginary part $\hat{F}(p,q)$ are related by the Cauchy–Riemann equations $F_p = \hat{F}_q$, $F_q = -\hat{F}_p$.

Accordingly, the function $G(p,q)$ can also be regarded as the real part of some analytic function of a complex variable:

$$g(s)=g(p+iq)=G(p,q)+i\hat{G}(p,q), \tag{54}$$

where the real part $G(p,q)$ and the imaginary part $\hat{G}(p,q)$ are calculated by the formulas

$$\begin{aligned} G(p,q) &= \lambda F(p,q)+v(p,q)F_p(p,q)+ \\ &+w(p,q)F_q(p,q)= \\ &=\lambda F(p,q)+v(p,q)F_p(p,q)- \\ &-w(p,q)\hat{F}_p(p,q), \\ \hat{G}(p,q) &= \lambda\hat{F}(p,q)+v(p,q)\hat{F}_p(p,q)+ \\ &+w(p,q)\hat{F}_q(p,q)= \\ &=\lambda\hat{F}(p,q)+v(p,q)\hat{F}_p(p,q)+ \\ &+w(p,q)F_p(p,q). \end{aligned} \tag{55}$$

We can verify that functions (55) are related by the Cauchy–Riemann equations:

$$G_p = \hat{G}_q,\ G_q = -\hat{G}_p.$$

Relations (55) for the real and the imaginary part of the function $g(s)$ are equivalent to an ordinary first-order differential equation for analytic functions of a complex variable [35]:

$$g(s)=\lambda f(s)+u(s)\frac{df(s)}{ds}, \tag{56}$$

where

$$u(s) = u(p + iq) = v(p,q) + iw(p,q).$$

Eq. (56), considered on the complex plane with respect to an unknown function $f(s)$ for a given function $g(s)$, has a solution

$$\begin{aligned} f(s) &= \left(C+\int_{s_0}^{s}\left[\frac{g(t)}{u(t)}\exp\left(\int_{s_0}^{t}\frac{\lambda d\tau}{u(\tau)}\right)\right]dt\right)\times \\ &\times\exp\left(-\int_{s_0}^{s}\frac{\lambda dt}{u(t)}\right), \end{aligned} \tag{57}$$

where c is an arbitrary complex constant.

It is important for Eq. (57) that the integral of the analytic function on the complex plane does not depend on the integration path (cuts eliminating the singular points of the integrands and ensuring that the resulting domain is simply connected may have to be added to the complex plane for this purpose), and the result of integration is analytic function [35]. For this reason, differential relation (51) is invertible for a subset of two-dimensional harmonic functions if $u(s) \neq 0$ (or for $u(s) = 0$ if $\lambda \neq 0$).

Therefore, the Thomson transforms generated for homogeneous harmonic functions (47)–(49) or (50) using relations (51) are Donkin operators, unless all coefficients in Eq. (51) vanish simultaneously.

Degenerate linear combinations

Until now, we have considered Donkin operators that can be applied to homogeneous harmonic functions of any degree, even though they explicitly contain the degree of homogeneity as a parameter in some cases. In this section we discuss whether degenerate cases exist that work only for one degree of homogeneity and cannot be generalized to an arbitrary degree of homogeneity.

Indeed, the degree of homogeneity of the formulas from the lists (5)–(14) and (15)–(24) can intersect for some values of *m*, and, therefore, these formulas can be combined in the same linear combination with constant coefficients. Such combinations may differ from general linear combinations considered in the section “Special case”. Let us give such values of *m*.

1. If $m = -1/2$, operators (5), (12)–(15), (22)–(24) have the same degree of homogeneity, equal to $-1/2$, (this case does not interest us, since groups of operators (5), (12)–(15), (22)–(24) are identically equal for $k = -1/2$).

2. If $m = 0$, operators (5), (12)–(14), (16)–(18) have the same degree of homogeneity, equal to zero, so that the linear combination of operators takes the form

$$L[U] = eU + U_x(cy - bz + fr) + \\ +U_y(az - cx + gr) + U_z(bx - ay + hr). \tag{58}$$

3. If $m = -1$, operators (5), (12)–(14), (19)–(21) have the same degree of homogeneity, equal to -1, so that the linear combination of operators takes the form

$$L[U] = U\left(e + \frac{fx + gy + hz}{r}\right) + \\ +U_x(cy - bz + fr) + U_y(az - cx + gr) + \\ +U_z(bx - ay + hr). \tag{59}$$

4. If $m = 0$, operators (6)–(8), (15), (22)–(24) have the same degree of homogeneity, equal to -1, so that the linear combination of operators takes the form

$$L[U] = \frac{eU}{r} + U_x\left(f + \frac{cy - bz}{r}\right) + \\ +U_y\left(g + \frac{az - cx}{r}\right) + U_z\left(h + \frac{bx - ay}{r}\right). \tag{60}$$

5. If $m = 1/2$, operators (6)–(8), (16)–(18) have the same degree of homogeneity, equal to $-1/2$, (this case does not interest us, since groups of operators (6)–(8) and (16)–(18) are identically equal for $k = 1/2$).

6. If $m = -1/2$, operators (6)–(8), (19)–(21) have the same degree of homogeneity, equal to $-3/2$, (this case also does not interest us, since groups of operators (6)–(8) and (19)–(21) are identically equal for $k = -1/2$).

7. If $m = -1$, operators (9)–(11), (15), (22)–(24) have the same degree of homogeneity, equal to zero, so that the linear combination of operators takes the form

$$\lambda = e, \\ v(p,q) = \frac{f - b}{2} - hp + cq + \\ +(a - g)pq - \frac{b + f}{2}(p^2 - q^2), \\ w(p,q) = \frac{a + g}{2} - cp - hq - \\ -(b + f)pq - \frac{a - g}{2}(p^2 - q^2) \tag{61}$$

8. If $m = -1/2$, operators (9)–(11), (16)–(18) have the same degree of homogeneity, equal to 1/2 (this case does not interest us, since groups of operators (9)–(11) and (16)–(18) are identically equal for $k = -1/2$).

9. If $m = -3/2$, operators (9)–(11), (19)–(21) have the same degree of homogeneity, equal to -1/2 (this case also does not interest us, since the group of operators (9)–(11) and (19)–(21) are identically equal for $k = -3/2$).

Expressions (58)–(61) correspond to the special case considered above, if we choose

$$\lambda = e, \\ v(p,q) = \frac{f - b}{2} - hp + cq + \\ +(a - g)pq - \frac{b + f}{2}(p^2 - q^2), \\ w(p,q) = \frac{a + g}{2} - cp - hq - \\ -(b + f)pq - \frac{a - g}{2}(p^2 - q^2) \tag{62}$$

(it is easy to verify that the functions $v(p,q)$ and $w(p,q)$ satisfy the Cauchy–Riemann equations $v_p = w_q$, $v_q = -w_p$). Therefore, operators (58)–(61) are Donkin operators with respect to homogeneous harmonic functions of the corresponding degrees for any choice of the constants *a*, *b*, *c*, *e*, *f*, *g*, *h*.

Conclusion

Considering all possible forms of Thomson differential transformations for three-dimensional homogeneous harmonic functions, we have found that any first-order Donkin operators are linear combinations with constant coefficients composed of basic Thomson differential formulas of the first order for homogeneous functions [2]. Naturally, this is not to say that basic Thomson differential formulas or their linear combinations are indeed Donkin operators. However, it was found in [3] that basic Thomson differential Eqs. (5)–(14), (15)–(24) are invertible, that is, they are Donkin operators.

We have established that linear combinations with constant coefficients, composed of basic Donkin operators (5)–(14), (15)–(24) corresponding to the same degree of homogeneity, are also Donkin operators. There are apparently no other first-order Donkin operators for three-dimensional homogeneous harmonic functions.

This is perhaps a rather strong statement that we should clarify. Adding Euler relation (S08) multiplied by an arbitrary function to any of Eqs. (5)–(14), (15)–(24) or to their linear combination, we obtain a new transforming formula with exactly the same properties. The reason for this is that these formulas are completely equivalent to basic Eqs. (5)–(14) and (15)–(24) or their linear combinations with constant coefficients in terms of their effect on three-dimensional homogeneous harmonic functions of a given degree.

To maintain experimental integrity, this function should be Euler-homogeneous with the corresponding degree of homogeneity; otherwise, the artificial additive would be too easy to isolate from the new expression. Such formulas may differ quite considerably from the list obtained earlier in the algebraic sense. For example, the operator

$$\begin{gathered} L[U] = AU_x(x,y,z) + \\ +BU_y(x,y,z) + CU_z(x,y,z), \\ A = a(2m+1)xz + b(2m+1)yz + \\ +c\left(-mx^2 - my^2 + (m+1)z^2\right), \\ B = a(2m+1)xy + \\ +b\left(-mx^2 + (m+1)y^2 - mz^2\right) + \\ +c(2m+1)yz, \\ C = a\left((m+1)x^2 - my^2 - mz^2\right) + \\ +b(2m+1)xy + c(2m+1)xz, \end{gathered} \tag{63}$$

with arbitrary constants a, b and c differs (in an algebraic sense) both from any of the previously obtained basic Eqs. (5)–(14) and (15)–(24) and from their linear combination with constant coefficients. However, this operator can be represented as

$$\begin{gathered} L[U] = \Big(a(2m+1)xz + b(2m+1)yz + \\ +c\left(-mx^2 - my^2 + (m+1)z^2\right)\Big)U_x + \\ +\Big(a(2m+1)xy + b\left(-mx^2 + (m+1)y^2 - mz^2\right) + \\ +c(2m+1)yz\Big)U_y + \\ +\Big(a\left((m+1)x^2 - my^2 - mz^2\right) + \\ +b(2m+1)xy + c(2m+1)xz\Big)U_z = \\ = (2m+1)(cx + by + az)\times \\ \times\left(xU_x + yU_y + zU_z - mU\right) + \\ +mc\left((2m+1)xU - r^2U_x\right) + \\ +mb\left((2m+1)yU - r^2U_y\right) + \\ +ma\left((2m+1)xU - r^2U_z\right). \end{gathered} \tag{64}$$

It follows from this identity that operator (63) is actually no different from a linear combination of operators (9)–(11) in its effect on homogeneous harmonic functions of degree m.

However, this relationship is not always obvious. Similar situations, when actually identical mathematical expressions are not identical in an algebraic sense, are considered, for example, in [36–39].

The calculations given in this paper were carried out using the Wolfram Mathematica software [40].

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