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# BASIC DONKIN'S DIFFERENTIAL OPERATORS FOR HOMOGENEOUS HARMONIC FUNCTIONS

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It has been shown that there are differential operators transforming the three-dimensional homogeneous harmonic functions into new three-dimensional ones. A characteristic feature of these operators is their invertibility: for any homogeneous harmonic function there is a homogeneous and harmonic prototype from which it can be obtained by applying the specified operator. The involved operators were called differential Donkin's operators by the authors. The paper provides a complete list of fundamental first-order Donkin's differential operators forming a linear basis of Thomson formulas for three-dimensional homogeneous harmonic functions.

**Keywords:** electrostatic field, magnetostatic field, scalar potential, homogeneous function, harmonic function

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# БАЗИСНЫЕ ДИФФЕРЕНЦИАЛЬНЫЕ ОПЕРАТОРЫ ДОНКИНА ДЛЯ ОДНОРОДНЫХ ГАРМОНИЧЕСКИХ ФУНКЦИЙ

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В работе показано, что существуют дифференциальные операторы, которые преобразуют трехмерные однородные гармонические функции в новые трехмерные однородные гармонические функции. Характерной чертой этих операторов, названных авторами дифференциальными операторами Донкина, является их обратимость: для любой однородной гармонической функции найдется однородный и гармонический прототип, из которого эту функцию можно получить, если применить указанный оператор. В работе приводится полный список дифференциальных операторов Донкина первого порядка, которые выступают в качестве линейного базиса для симметризованных формул Томсона, используемых при генерировании трехмерных однородных гармонических функций.

**Ключевые слова:** электростатическое поле, магнитостатическое поле, скалярный потенциал, однородная функция, гармоническая функция

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## Problem statement

Euler-homogeneous electric and magnetic fields are a useful tool for synthesizing electron-optical and ion-optical systems of a special type [1–6]. The electric field strength **E** and/or the magnetic flux density **B** for Euler-homogeneous fields should satisfy not only the Maxwell equations for the electromagnetic field but also the homogeneity identity:

$$\forall\lambda>0:\ \mathbf{E}(\lambda x,\lambda y,\lambda z)\equiv\lambda^{k-1}\mathbf{E}(x,y,z),$$
$$\forall\lambda>0:\ \mathbf{B}(\lambda x,\lambda y,\lambda z)\equiv\lambda^{k-1}\mathbf{B}(x,y,z),\qquad(1)$$

where $k$ is the degree of field homogeneity (not necessarily an integer).

The trajectories of charged particles in Euler-homogeneous electrostatic and magnetostatic fields obey Golikov's principle of similarity of trajectories [7, 8]. The unique optical properties of devices controlling the motion of charged particles and using Euler-homogeneous electric and magnetic fields follow from this principle [1–5, 9–16].

Given a non-zero degree of homogeneity, Euler-homogeneous electric or magnetic fields obeying identity (1) are characterized by a scalar electric or magnetic potential in the form of a scalar harmonic function $U(x,y,z)$, which are homogeneous (more precisely, positively homogeneous, i.e. with $\lambda > 0$) in the sense given to this term in classical mathematical analysis [17, 18]:

$$\forall\lambda>0:\ U(\lambda x,\lambda y,\lambda z)\equiv\lambda^{k}U(x,y,z),\qquad(2)$$

where $k$ is the degree of homogeneity of the function, coinciding with the degree of homogeneity of the electric or magnetic field, as determined by identities (1).

In the most general case, the scalar potential $U$ for fields with zero degree of homogeneity has the form:

$$U(x,y,z)=U_0(x,y,z)+C\ln(z+r),\qquad(3)$$

where $r=\sqrt{x^2+y^2+z^2}$ (from now on), $U_0$ is a homogeneous harmonic function of degree zero, $C$ is an arbitrary constant.

If $C\neq 0$, expression (3) is no longer an Euler-homogeneous function, although the gradient of function (3) obeys homogeneity conditions (1) for an electric or a magnetic field. The issue of whether scalar and/or vector potentials are homogeneous for Euler-homogeneous fields is explored in detail in [19].

Golikov's studies [3, 20, 21] clearly demonstrate that analytical methods and, in particular, analytical expressions for potentials of electric and magnetic fields are a convenient and effective tool for synthesizing new electronic and ion-optical systems. Additional measures have to be takes for obtaining scalar potentials for homogeneous electric and magnetic fields, since there are, in a sense, "far less" such functions than ordinary harmonic ones (see Note 1 at the end of the paper). Some useful analytical expressions for Euler-homogeneous harmonic functions, which can be used as scalar potentials, are given in [22–29]. However, if we confine ourselves to considering these analytical expressions in optimizing new electron and ion-optical systems, we run the risk of missing any truly optimal solutions.

The problem of calculating homogeneous harmonic functions with integer degrees of homogeneity has been solved completely. More precisely, a new three-dimensional homogeneous harmonic function of degree $k-1$ can be obtained from any three-dimensional homogeneous harmonic function $U(x,y,z)$ of degree $k$ by differentiating with respect to one of the space variables $x$, $y$ or $z$. Then we obtain a chain of homogeneous harmonic functions with successively decreasing degrees of homogeneity for integer degrees. Using the Thomson formula (Kelvin transform) [30–41], which has the form

$$V(x,y,z)=\frac{1}{r}U\left(\frac{x}{r^2},\frac{y}{r^2},\frac{z}{r^2}\right),\qquad(4)$$

homogeneous harmonic functions $V$ with positive integer degrees $(-k-1)$ are obtained from homogeneous harmonic functions $U$ with negative integer degrees $k$. Finally, there are explicit Donkin formulas for homogeneous harmonic functions of degree zero and degree $-1$, [1, 2, 11, 24, 42–45]:

$$V_0(x,y,z)=H\left(\frac{x}{z+r},\frac{y}{z+r}\right),\qquad(5)$$

$$V_{-1}(x,y,z)=\frac{1}{r}H\left(\frac{x}{z+r},\frac{y}{z+r}\right);\qquad(6)$$

they establish a one-to-one correspondence between the solutions $H(p,q)$ of the two-dimensional Laplace equation

$$H_{pp}+H_{qq}=0$$

and three-dimensional homogeneous harmonic functions with homogeneity degrees 0 and $-1$.

Since the common factor $1/r^2$ for homogeneous functions can be taken out of the sign of the function, formula (4) can be rewritten in a more convenient form:

$$V(x,y,z)=r^{-2k-1}U(x,y,z). \qquad (7)$$

Formula (7) is called the Thomson formula for homogeneous functions. Evidently, expression (7) is an Euler-homogeneous function of degree $(-k-1)$ if $U$ is an Euler-homogeneous function of degree $k$. However, straightforward proof (that does not reference the general Thomson formula (4)) that expression (7) is harmonic provided that the function $U$ is harmonic requires some extra effort. In particular, this proof can be found in treatise [32] (see Appendix B to Chapter 1).

Let us now provide the historical background for our discussion. William Thomson, Lord Kelvin, was a British physicist known for his outstanding achievements in pure and applied science. Formula (2) is called either the Thomson formula or the Kelvin transform in literature, depending on the traditions held by the researcher. It appears that the term “Thomson formula” is exclusively used for formula (3) in general mathematical literature, typically with the clarification “for homogeneous functions”. We have introduced the combined term “Thomson transform” for linear differential-algebraic operators of a general form, transforming harmonic functions into new harmonic functions.

The above process for obtaining general formulas for homogeneous harmonic functions (consisting of differentiating Donkin formulas (5) or (6) with respect to the variables $x$, $y$ or $z$ and subsequently applying the Thomson formula (4) or (7)) yields a wide variety of new homogeneous harmonic functions with integer degrees of homogeneity. An undoubted advantage of this technique is that it allows to obtain any homogeneous harmonic function of an integer degree. It is fairly simply (though not entirely obvious) to substantiate that generating three-dimensional homogeneous harmonic functions with integer degrees of homogeneity is a complete process.

Indeed, the theorem on differentiation of homogeneous harmonic functions [44, 46] ensures that there is a homogeneous harmonic prototype for any homogeneous harmonic function, whose degree is greater by unity and from which this function can be obtained by differentiation with respect to any of the space variables $x$, $y$ or $z$. The Thomson formula has a similar property of invertibility: there is a homogeneous and harmonic prototype for any homogeneous harmonic function from which the function can be obtained by applying transformation (4) or (7). This statement is almost obvious, since the Thomson transform, repeated twice, leads to the original homogeneous harmonic function.

Finally, Donkin formulas (5), (6) exhaustively describe three-dimensional homogeneous harmonic functions with degrees of homogeneity 0 and −1; the real and imaginary parts of analytical functions of a complex variable can be used as a source of two-dimensional harmonic functions for Donkin formulas [47–50]. As a result, any three-dimensional homogeneous harmonic function of integer degree has a prototype of the form (5) or (6) which is, as a rule, not the only one.

Practically speaking, it is very important that there is a constructive process making it theoretically possible to search through all homogeneous harmonic functions with integer degrees of homogeneity using two-dimensional harmonic functions as the source material (see Note 2 at the end of the paper). An algorithm for obtaining homogeneous harmonic functions with integer degrees of homogeneity is described in more detail in [24, 44]. Unfortunately, the issue of obtaining and providing a complete description for homogeneous harmonic functions with non-integer degrees of homogeneity remains open at present.

Differentiation with respect to the variables $x$, $y$, and $z$ is not the only transformation with the required properties. Let us call a linear differential operator the Donkin operator (by the name of the discoverer of this class of transformations [42–45]) under the following conditions:

a) the operator converts an arbitrary homogeneous harmonic function into a new homogeneous harmonic function, possibly with a different degree of homogeneity;

b) there is a prototype function for any homogeneous harmonic function (also homogeneous and harmonic, but not necessarily the only one) from which this homogeneous harmonic function can be obtained using the operator in question.

The identity transformation $L[U] = U$ (possibly involving an arbitrary non-zero factor) is a trivial and not particularly interesting example. The theorem on differentiation of homogeneous harmonic functions [44, 46] states that the differentiation operators

$$L[U] = U_x,\ L[U] = U_y,\ L[U] = U_z,$$

or, in general, the operator

$$L[U] = aU_x + bU_y + cU_z,$$

($a$, $b$, $c$ are constants, not all are equal to zero) are certainly Donkin operators.

The goal of this study has consisted in finding an exhaustive list of first-order Donkin operators for three-dimensional homogeneous harmonic functions, continuing the work started in [40, 41].

### Example of a nontrivial Donkin operator

A comprehensive list of all three-dimensional homogeneous harmonic functions with integer non-negative orders of homogeneity was apparently given for the first time in [42, 43]. The author used a linear differential operator of a special form for this purpose; let us consider this operator in this section.

Let a homogeneous harmonic function $U(x,y,z)$ be given with a degree of homogeneity equal to $m$. The procedure for constructing a new homogeneous harmonic function $V(x,y,z)$ with a degree of homogeneity equal to $m + 1$ is as follows, according to [42, 43].

A one-to-one mapping from Cartesian variables $x$, $y$, $z$ to spherical ones $r$, $\theta$, $\varphi$, i.e.,

$$\begin{cases} x = r\sin\theta\cos\varphi, \\ y = r\sin\theta\sin\varphi, \\ z = r\cos\theta \end{cases} \Leftrightarrow \quad \Leftrightarrow \begin{cases} r = \sqrt{x^2+y^2+z^2}, \\ \theta = arctg\left(\sqrt{x^2+y^2}/z\right), \\ \varphi = arctg\left(y/x\right), \end{cases} \tag{8}$$

is used to write the Euler-homogeneous function $U$ in canonical form:

$$U(x,y,z) = r^m u(\theta,\varphi). \tag{9}$$

A spherical function $u(\theta,\varphi)$, given on a unit sphere, is uniquely defined by the predetermined homogeneous function $U$. Since the function $U$ *is* harmonic (satisfying the three-dimensional Laplace equation), the function $u(\theta,\varphi)$ should satisfy the equation

$$\frac{\partial^2 u(\theta,\varphi)}{\partial\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial u(\theta,\varphi)}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 u(\theta,\varphi)}{\partial\varphi^2} + m(m+1)u(\theta,\varphi) = 0. \tag{10}$$

It follows then, in particular, that the function

$$U^*(x,y,z) = r^{-m-1}u(\theta,\varphi)$$

is also harmonic (see Note 3 at the end of the paper).

Transition from the spherical function $u(\theta,\varphi)$ to a new spherical function $v(\theta,\varphi)$ is made using the transformation

$$v(\theta,\varphi) = \sin\theta\frac{\partial u(\theta,\varphi)}{\partial\theta} + (m+1)\cos\theta\cdot u(\theta,\varphi). \tag{11}$$

We can verify that the resulting function $v(\theta,\varphi)$ satisfies the differential equation

$$\frac{\partial^2 v(\theta,\varphi)}{\partial\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial v(\theta,\varphi)}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 v(\theta,\varphi)}{\partial\varphi^2} + (m+1)(m+2)v(\theta,\varphi) = 0, \tag{12}$$

since the function $u(\theta,\varphi)$ satisfies Eq. (10).

The new function $V$ is found by the formula

$$V(x,y,z) = r^{m+1}v(\theta,\varphi). \tag{13}$$

Such a function $V$ is not only Euler-homogeneous with the degree of homogeneity $m + 1$, but should also satisfy the three-dimensional Laplace equation, since the function $v(\theta,\varphi)$ satisfies Eq. (12).

If we now summarize the construction procedure described here, it turns out that the new function $V$ is expressed in terms of the old function $U$ using a linear differential operator of the first order, which explicitly depends on the degree of homogeneity of the harmonic function:

$$V = L[U] = xz\frac{\partial U}{\partial x} + yz\frac{\partial U}{\partial y} - (x^2+y^2)\frac{\partial U}{\partial z} + (m+1)zU. \tag{14}$$

Given that the function $U$ satisfies the Euler differential relation for homogeneous functions of degree $m$ [17, 18], i.e.,

$$x\frac{\partial U}{\partial x} + y\frac{\partial U}{\partial y} + z\frac{\partial U}{\partial z} - mU = 0, \tag{15}$$

operator (14) can be transformed to the form

$$V(x,y,z) = -(x^2+y^2+z^2)\frac{\partial U(x,y,z)}{\partial z} + (2m+1)zU(x,y,z). \tag{16}$$

Formula (16) transforms homogeneous functions into new homogeneous functions with a degree of homogeneity greater by unity, since the partial derivatives of Euler-homogeneous functions are also homogeneous [17, 18].

If the function $U$ *is* harmonic, the function $V$ given by transformation (16) is preserved harmonic by the identity

$$V_{xx}+V_{yy}+V_{zz}\equiv(2m+1)z\left(U_{xx}+U_{yy}+U_{zz}\right)-\\ -r^2\left(U_{xxz}+U_{yyz}+U_{zzz}\right)- \qquad (17)\\ -4\left(xU_{xz}+yU_{yz}+zU_{zz}-(m-1)U_z\right).$$

Since the function $U$ should satisfy both the Laplace equation and the Euler relation (15), and also additional equations of a higher order (they are obtained by differentiating the Laplace equation and the Euler relation with respect to the variable $z$), the right-hand side of equality (17) becomes zero.

**Theorem 1.** *Operator (16) is invertible, i.e., there is a prototype in the form of a homogeneous harmonic function U of degree m for any homogeneous harmonic function of degree V(m* + 1) *such that the functions V and U are related by condition (16).*

P r o o f. Thomson transform (7) allows to find for a homogeneous harmonic function $U$ of degree $m$ such a homogeneous harmonic function $\hat{U}$ of degree $(-m-1)$ that ensures the equality

$$U(x,y,z)=r^{2m+1}\hat{U}(x,y,z). \qquad (18)$$

Similarly, there is a homogeneous harmonic function $\hat{V}$ of degree $(-m-2)$ for a homogeneous harmonic function $V$ of degree $(m+1)$, which ensures the equality

$$V(x,y,z)=r^{2m+3}\hat{V}(x,y,z). \qquad (19)$$

If we substitute equalities (18) and (19) into expression (16) and simplify the resulting expression, we obtain the following condition:

$$\hat{V}(x,y,z)=-\frac{\partial\hat{U}(x,y,z)}{\partial z}. \qquad (20)$$

According to the theorem on differentiation of three-dimensional homogeneous harmonic functions [44, 46], such a three-dimensional homogeneous harmonic prototype function $\hat{U}$ can be found for a given arbitrary homogeneous harmonic function $\hat{V}$ which ensures that relation (20) holds true. Then the homogeneous harmonic function $U$, which is obtained from the function $\hat{U}$ using the Thomson transform (7), ensures that equality (16) holds true.

Therefore, linear differential operator (16) is invertible on the subset of homogeneous harmonic functions of degree $m$, that is, it is a Donkin operator.

Theorem 1 is proved.

Donkin operator (16) is a complete equivalent of the differentiation operation in terms of generating a complete set of homogeneous harmonic functions with integer degrees of homogeneity. Indeed, since it is known that all homogeneous harmonic functions of degree zero can be obtained using Donkin formula (5), by successively applying operators of the form (16) to expressions (5), we can obtain all homogeneous harmonic functions with degrees of homogeneity 1, 2,....

The invertibility of differential operator (16) ensures that no homogeneous harmonic function is missed. However, since the prototype $U$ is not uniquely determined by the function $V$, most homogeneous harmonic functions occur more than once during generation (namely, the same homogeneous harmonic function can be obtained very different prototype functions).

A complete set of homogeneous harmonic function pairs with negative integer degrees −1, −2,... can be obtained using the Thomson transforms (4) or (7) from a complete set of homogeneous harmonic functions with integer degrees 0, 1, 2,.... Thus, operator (16), like differentiation of homogeneous harmonic functions with respect to $x$, $y$ or $z$ [44, 46], yields an exhaustive representation of homogeneous harmonic functions with integer degrees of homogeneity.

However, as noted above, compiling an exhaustive representation for the complete set of homogeneous harmonic functions with non-integer degrees of homogeneity remains an important problem that is yet to be resolved.

## Thomson differential formulas of the first order

Differential equivalents of the first order of the Thomson formula (4) are given in [40] for three-dimensional harmonic functions. These formulas transform three-dimensional harmonic functions into new three-dimensional harmonic functions; the form of the transforming expressions is deliberately chosen so that substituting Euler-homogeneous functions into them produces new Euler-homogeneous functions, possibly with a different degree of homogeneity. This list of formulas is as follows:

$$V(x,y,z) = U(x,y,z), \tag{21}$$

$$V(x,y,z) = U_x(x,y,z), \tag{22}$$

$$V(x,y,z) = U_y(x,y,z), \tag{23}$$

$$V(x,y,z) = U_z(x,y,z), \tag{24}$$

$$\begin{gathered} V(x,y,z) = xU(x,y,z) + \\ +(x^2 - y^2 - z^2)U_x(x,y,z) + \\ +2xyU_y(x,y,z) + 2xzU_z(x,y,z), \end{gathered} \tag{25}$$

$$\begin{gathered} V(x,y,z) = yU(x,y,z) + \\ +2xyU_x(x,y,z) + \\ +(-x^2 + y^2 - z^2)U_y(x,y,z) + \\ +2yzU_z(x,y,z), \end{gathered} \tag{26}$$

$$\begin{gathered} V(x,y,z) = zU(x,y,z) + \\ +2xzU_x(x,y,z) + \\ +2yzU_y(x,y,z) + \\ +(-x^2 - y^2 + z^2)U_z(x,y,z), \end{gathered} \tag{27}$$

$$\begin{gathered} V(x,y,z) = xU_x(x,y,z) + \\ +yU_y(x,y,z) + zU_z(x,y,z), \end{gathered} \tag{28}$$

$$V(x,y,z) = yU_x(x,y,z) - xU_y(x,y,z), \tag{29}$$

$$V(x,y,z) = zU_x(x,y,z) - xU_z(x,y,z), \tag{30}$$

$$V(x,y,z) = zU_y(x,y,z) - yU_z(x,y,z). \tag{31}$$

This list does not contain the formulas given in [40], obtained using the Thomson transform in form (4) for formulas (21)–(31), since such a transformation changes the arguments of the function and the differential operator no longer has a canonical form. However, the following operators are added to the list of formulas (21)–(31) upon applying the Thomson transform in the form (7) to the operators (21)–(31):

$$V(x,y,z) = \frac{1}{r^{2m+1}}U(x,y,z), \tag{32}$$

$$V(x,y,z) = \frac{1}{r^{2m+1}}U(x,y,z), \tag{33}$$

$$V(x,y,z) = \frac{1}{r^{2m-1}}U_x(x,y,z), \tag{34}$$

$$V(x,y,z) = \frac{1}{r^{2m-1}}U_z(x,y,z), \tag{35}$$

$$\begin{gathered} V(x,y,z) = \frac{x}{r^{2m+3}}U(x,y,z) + \\ +\frac{x^2 - y^2 - z^2}{r^{2m+3}}U_x(x,y,z) + \\ +\frac{2xy}{r^{2m+3}}U_y(x,y,z) + \frac{2xz}{r^{2m+3}}U_z(x,y,z), \end{gathered} \tag{36}$$

$$\begin{gathered} V(x,y,z) = \frac{y}{r^{2m+3}}U(x,y,z) + \\ +\frac{2xy}{r^{2m+3}}U_x(x,y,z) + \\ +\frac{-x^2 + y^2 - z^2}{r^{2m+3}}U_y(x,y,z) + \\ +\frac{2yz}{r^{2m+3}}U_z(x,y,z), \end{gathered} \tag{37}$$

$$\begin{gathered} V(x,y,z) = \frac{z}{r^{2m+3}}U(x,y,z) + \\ +\frac{2xz}{r^{2m+3}}U_x(x,y,z) + \frac{2yz}{r^{2m+3}}U_y(x,y,z) + \\ +\frac{-x^2 - y^2 + z^2}{r^{2m+3}}U_z(x,y,z), \end{gathered} \tag{38}$$

$$\begin{gathered} V(x,y,z) = \frac{x}{r^{2m+1}}U_x(x,y,z) + \\ +\frac{y}{r^{2m+1}}U_y(x,y,z) + \frac{z}{r^{2m+1}}U_z(x,y,z), \end{gathered} \tag{39}$$

$$\begin{gathered} V(x,y,z) = \frac{y}{r^{2m+1}}U_x(x,y,z) - \\ -\frac{x}{r^{2m+1}}U_y(x,y,z), \end{gathered} \tag{40}$$

$$\begin{gathered} V(x,y,z) = \frac{z}{r^{2m+1}}U_x(x,y,z) - \\ -\frac{x}{r^{2m+1}}U_z(x,y,z), \end{gathered} \tag{41}$$

$$\begin{gathered} V(x,y,z) = \frac{z}{r^{2m+1}}U_y(x,y,z) - \\ -\frac{y}{r^{2m+1}}U_z(x,y,z). \end{gathered} \tag{42}$$

Unlike the differential operators obtained from (21)–(31) after the Thomson transform in the form (4), these formulas preserve the arguments of the function $U$ unchanged. However, now the operators (32)–(42) depend on the degree of homogeneity of the function in explicit form and, in addition, generate new harmonic functions only from other homogeneous harmonic functions. On the other hand, formulas (21)–(31) do not explicitly depend on the degree of homogeneity of the transformed function, and also produce harmonic functions from arbitrary three-dimensional harmonic functions that are not necessarily Euler-homogeneous.

**Theorem 2.** *First-order linear differential operators (21)–(27), (29)–(38), (40)–(42), transforming homogeneous harmonic functions into new homogeneous harmonic functions, are invertible on the set of homogeneous harmonic functions of degree m.*

Proof. Formulas (32)–(42) inherit their invertibility and/or non-invertibility from formulas (21)–(31), since the Thomson transform (7) is certainly invertible. Therefore, it is sufficient to either prove or refute the invertibility only for formulas (21)–(31).

The invertibility of operator (21) is trivial, so, in accordance with the definition given earlier, the identity transformation

$$V(x,y,z) = U(x,y,z)$$

is a special, although completely useless, case of a Donkin operator. The invertibility of operators (22)–(24) follows from the theorem on differentiation of homogeneous harmonic functions, a rigorous proof for which is given in [46].

Operators (25)–(27) are obtained by differentiating the Thomson formula (4) with respect to the variables $x$, $y$, and $z$, followed by applying another Thomson transform to return the arguments of the function to the previous form. Let us now perform these transformations in reverse order.

1. A given homogeneous harmonic function $V(x,y,z)$ has a homogeneous and harmonic prototype $V^*(x,y,z)$, from which it can be obtained by means of transformation (4).

2. A homogeneous harmonic function $V^*(x,y,z)$ has a homogeneous and harmonic prototype $U^*(x,y,z)$, from which it can be obtained by differentiation: either $V^* = U^*_x$ or $V^* = U^*_y$ or $V^* = U^*_z$ [44, 46].

3. A homogeneous harmonic function $U^*(x,y,z)$ has a homogeneous and harmonic prototype $U(x,y,z)$, from which it can be obtained by means of transformation (4).

The homogeneous harmonic function $U(x,y,z)$ turns out to be a prototype, from which, by transforming (25), (26), or (27), we obtain the given homogeneous harmonic function $V(x,y,z)$ (transformations (25)–(27) are a superposition of transformations (1)–(3), made in this exact order, which can be easily verified by directly calculating this superposition).

For this reason, transformations (25)–(27) are invertible and, therefore, are Donkin operators in accordance with the definition given earlier.

The remaining task is to deal with transformations (28)–(31). For example, let us consider transformation (29):

$$V(x,y,z) = yU_x(x,y,z) - xU_y(x,y,z). \quad (43)$$

By replacing the Donkin coordinates [1, 2, 11, 24], namely,

$$\begin{cases} p = \dfrac{x}{z+\sqrt{x^2+y^2+z^2}}, \\ q = \dfrac{y}{z+\sqrt{x^2+y^2+z^2}}, \\ r = \sqrt{x^2+y^2+z^2}, \end{cases} \Leftrightarrow \quad (44)$$

$$\Leftrightarrow \begin{cases} x = \dfrac{2pr}{1+p^2+q^2}, \\ y = \dfrac{2qr}{1+p^2+q^2}, \\ z = \pm\dfrac{r\left(1-p^2-q^2\right)}{1+p^2+q^2}, \end{cases}$$

the functions $U$ and $V$, Euler-homogeneous with the degree of homogeneity $m$, can be written as

$$U(x,y,z) = r^m F\left(\frac{x}{z+r}, \frac{y}{z+r}\right), \quad (45)$$

$$V(x,y,z) = r^m G\left(\frac{x}{z+r}, \frac{y}{z+r}\right). \quad (46)$$

The necessary and sufficient condition for the functions $U(x,y,z)$ and $V(x,y,z)$ to be harmonic (satisfying the Laplace equation) is that the functions $F(p,q)$ and $G(p,q)$ satisfy the differential equations

$$\frac{\partial^2 F(p,q)}{\partial p^2}+\frac{\partial^2 F(p,q)}{\partial q^2}+ +\frac{4m(m+1)}{\left(1+p^2+q^2\right)^2}F(p,q)=0, \quad (47)$$

$$\frac{\partial^2 G(p,q)}{\partial p^2}+\frac{\partial^2 G(p,q)}{\partial q^2}+ +\frac{4m(m+1)}{\left(1+p^2+q^2\right)^2}G(p,q)=0. \quad (48)$$

With this substitution, Eq. (43) takes the form

$$q\frac{\partial F(p,q)}{\partial p}-p\frac{\partial F(p,q)}{\partial q}=G(p,q). \quad (49)$$

The functions $F$ and $G$ in polar coordinates $p = s\cos\varphi$, $q = s\sin\varphi$ take the form

$$F(p,q)=F_0\left(\operatorname{arctg}(q/p),\sqrt{p^2+q^2}\right), \quad (50)$$

$$G(p,q)=G_0\left(\operatorname{arctg}(q/p),\sqrt{p^2+q^2}\right), \quad (51)$$

and Eqs. (47), (48), (49) are written as

$$\frac{\partial^2 F_0(\varphi,s)}{\partial\varphi^2}+s^2\frac{\partial^2 F_0(\varphi,s)}{\partial s^2}+ +s\frac{\partial F_0(\varphi,s)}{\partial s}+\frac{4m(m+1)s^2}{\left(1+s^2\right)^2}F_0(\varphi,s)=0, \quad (52)$$

$$\frac{\partial^2 G_0(\varphi,s)}{\partial\varphi^2}+s^2\frac{\partial^2 G_0(\varphi,s)}{\partial s^2}+ +s\frac{\partial G_0(\varphi,s)}{\partial s}+\frac{4m(m+1)s^2}{\left(1+s^2\right)^2}G_0(\varphi,s)=0. \quad (53)$$

$$\frac{\partial F_0(\varphi,s)}{\partial\varphi}+G_0(\varphi,s)=0. \quad (54)$$

It follows from Eq. (54) that the general solution of the problem has the form

$$F_0(\varphi,s)=H(s)-\int_{\varphi_0}^{\varphi}G_0(\xi,s)\,d\xi, \quad (55)$$

where the free function $H(s)$ should be chosen so that Eq. (52) holds true.

After substitution (55), Eq. (52) takes the form

$$s^2\frac{d^2H(s)}{ds^2}+s\frac{dH(s)}{ds}+ +\frac{4m(m+1)s^2}{\left(1+s^2\right)^2}H(s)=H_0(\varphi,s), \quad (56)$$

$$H_0(\varphi,s)=\frac{4m(m+1)s^2}{\left(1+s^2\right)^2}\int_{\varphi_0}^{\varphi}G_0(\xi,s)\,d\xi+ +\frac{\partial G_0(\varphi,s)}{\partial\varphi}+s\int_{\varphi_0}^{\varphi}\frac{\partial G_0(\xi,s)}{\partial s}d\xi+ +s^2\int_{\varphi_0}^{\varphi}\frac{\partial^2 G_0(\xi,s)}{\partial s^2}d\xi. \quad (57)$$

Eq. (56) has a solution provided that the function $H_0(\varphi,s)$ on the right-hand side should not depend on the variable $\varphi$.

This is indeed true:

$$\frac{\partial H_0(\varphi,s)}{\partial\varphi}=\frac{4m(m+1)s^2}{\left(1+s^2\right)^2}G_0(\varphi,s)+ +\frac{\partial^2 G_0(\varphi,s)}{\partial\varphi^2}+s\frac{\partial G_0(\varphi,s)}{\partial s}+ +s^2\frac{\partial^2 G_0(\varphi,s)}{\partial s^2}=0, \quad (58)$$

since, in accordance with the initial condition, the function $G_0(\varphi,s)$ should satisfy Eq. (53).

A suitable solution $H(s)$ is then found from ordinary differential Eq. (56), where the function

$$H_0(\varphi_0,s)=\partial G(\varphi_0,s)/\partial\varphi,$$

independent of the variable $\varphi$, can be used as the right-hand side.

Next, the function $F_0(\varphi,s)$ is found from relation (55) and the function $F(p,q)$ from equality (50). Since both Eq. (47) and Eq. (49) are satisfied for the obtained function $F(p,q)$, the homogeneous function $U(x,y,z)$ given by formula (47) is harmonic and satisfies relation (43). Consequently, the transformation of homogeneous harmonic functions (29) is invertible, and linear differential operator (29) is the Donkin operator. Obviously, Donkin's operators are also transformations (30) and (31) (interchanging space coordinates is an invertible transformation).

Thus, differential operators (21)–(27), (29)–(31) and, accordingly, differential operators (32)–(38), (40)–(42) are invertible for a subset of homogeneous harmonic functions of degree $m$.

Theorem 2 is proved.

Operator (28) and, accordingly, operator (39) are considered separately. It follows from Euler relation (15) for homogeneous functions that the action of operator (28) on a homogeneous harmonic function $U$ *is* equivalent to multiplying it by the degree of homogeneity $m$, so operator (28) cannot be used to obtain fundamentally new homogeneous harmonic functions. Formally speaking, this operator is invertible with $m \neq 0$, and therefore it should be assumed to be a Donkin operator. However, operator (28) produces zero with $m = 0$, and if only a homogeneous harmonic function $V$ of degree zero is not equal to zero, it does not and cannot have a homogeneous and harmonic prototype $U$ of degree zero from which this function can be obtained by applying operator (28).

Nevertheless, when $V$ is a homogeneous harmonic function of zero degree, operator (28) has harmonic, but not homogeneous prototype functions for $V$. Such functions have the form

$$U(x,y,z) = U_0(x,y,z) + V(x,y,z)\ln(z + r),$$

where $U_0(x,y,z)$ is a homogeneous function of zero degree, chosen so as to preserve the functions $U$ harmonic.

Indeed, after we substitute

$$V(x,y,z) = W\left(\frac{x}{z+r}, \frac{y}{z+r}\right),$$

$$U_0(x,y,z) = H\left(\frac{x}{z+r}, \frac{y}{z+r}\right),$$

which can be used because $U_0(x,y,z)$ and $V(x,y,z)$ are homogeneous functions of zero degree, the condition

$$xU_x + yU_y + zU_z = V$$

is satisfied automatically, and for the function $U(x,y,z)$ to be harmonic, given that the function $W(p,q)$ satisfies the two-dimensional Laplace equation as the function $V(x,y,z)$ is harmonic, the function $H(p,q$ should satisfy the two-dimensional Poisson equation:

$$\frac{\partial^2 H}{\partial p^2} + \frac{\partial^2 H}{\partial q^2} = \frac{4}{1 + p^2 + q^2}\left(p\frac{\partial W}{\partial p} + q\frac{\partial W}{\partial q}\right),$$

which obviously has solutions.

### Simplified form for Donkin operators

If we assume that the function $U$ is a homogeneous function of degree $m$ obeying the Euler differential relation (15), expressions (25)–(27) and (36)–(38) are simplified:

$$V(x,y,z) = (2m+1)xU(x,y,z) - -r^2U_x(x,y,z), \quad (59)$$

$$V(x,y,z) = (2m+1)yU(x,y,z) - -r^2U_y(x,y,z), \quad (60)$$

$$V(x,y,z) = (2m+1)zU(x,y,z) - -r^2U_z(x,y,z), \quad (61)$$

$$V(x,y,z) = \frac{(2m+1)x}{r^{2m+3}}U(x,y,z) - -\frac{1}{r^{2m+1}}U_x(x,y,z), \quad (62)$$

$$V(x,y,z) = \frac{(2m+1)y}{r^{2m+3}}U(x,y,z) - -\frac{1}{r^{2m+1}}U_y(x,y,z), \quad (63)$$

$$V(x,y,z) = \frac{(2m+1)z}{r^{2m+3}}U(x,y,z) - -\frac{1}{r^{2m+1}}U_z(x,y,z), \quad (64)$$

where $r = \sqrt{x^2 + y^2 + z^2}$.

Operators (28) and (39) are not included in this list. The simplified operator (28) takes the form

$$V(x, y, z) = mU(x, y, z)$$

and thus coincides with the operator (21) (the identity transformation) up to a constant factor.

Similarly, operator (39) takes the form

$$V(x,y,z) = mU(x,y,z)/r^{2m+1}$$

and thus coincides with operator (32) (Thomson transform (7)) up to a constant factor.

Despite the obvious differences, Eqs. (59)–(64) are essentially already well-known Donkin operators, only written in a different form. In particular, Eq. (61) coincides with operator (16), which was considered in the section “Example of a non-trivial Donkin operator”. Moreover, operators (59)–(64) are algebraically different from operators (25)–(27) and (36)–(38) that generated them, so their action on arbitrary functions $U(x,y,z)$ is generally different. However, applying operators (25)–(27), (36)–(38) and operators (59)–(61), (62)–(64) to homogeneous functions of degree $m$ leads to the same result.

## Conclusion

Analysis of all possible representations of Thomson differential formulas for three-dimensional homogeneous harmonic functions, carried out in [40, 41] has revealed there are no other Donkin operators for three-dimensional homogeneous harmonic functions up to linear combinations with constant coefficients and, in particular, up to multiplication by a constant.

This is perhaps a rather strong statement that we should clarify. For example, other relations, different from those obtained earlier, can be formulated for linear combinations with constant coefficients composed of basic relations (21)–(31) and (32)–(42), with the same degree of homogeneity of the transformed function. Another possibility is using simplified expressions (59)–(64) for this purpose. We plan to dedicate a separate publication to study of invertible linear combinations composed of basic Donkin operators (21)–(31) and (32)–(42).

At the same time, in addition to simplified expressions (59)–(64), there are other equivalent forms of Donkin operators, for example, those given in the previous section; after all, adding Euler relation (16) multiplied by an arbitrary function to each of formulas (21)–(31) or (32)–(42), we obtain a new transforming formula with the same properties. Notably, to maintain experimental integrity, this function should be Euler-homogeneous with the corresponding degree of homogeneity; otherwise, the artificial additive would be too easy to isolate from the new expression.

Let us consider, for example, the operator

$$\begin{gathered} L[U] = AU_x(x,y,z) + \\ +BU_y(x,y,z) + CU_z(x,y,z), \\ A = a(2m+1)xz + b(2m+1)yz + \\ +c\left(-mx^2 - my^2 + (m+1)z^2\right), \\ B = a(2m+1)xy + \\ +b\left(-mx^2 + (m+1)y^2 - mz^2\right) + c(2m+1)yz, \\ C = a\left((m+1)x^2 - my^2 - mz^2\right) + \\ +b(2m+1)xy + c(2m+1)xz, \end{gathered} \tag{65}$$

with arbitrary constants $a$, $b$ and $c$ (the same operator was used in our study [41]).

From an algebraic standpoint, this operator differs both from any of the previously obtained basic formulas (21)–(31), and from their linear combination with constant coefficients but it can be represented in the form

$$\begin{gathered} L[U] = \Big(a(2m+1)xz + b(2m+1)yz + \\ +c\left(-mx^2 - my^2 + (m+1)z^2\right)\Big)U_x + \\ +\Big(a(2m+1)xy + b\left(-mx^2 + (m+1)y^2 - mz^2\right) + \\ +c(2m+1)yz\Big)U_y + \\ +\Big(a\left((m+1)x^2 - my^2 - mz^2\right) + \\ +b(2m+1)xy + c(2m+1)xz\Big)U_z = \\ = (2m+1)(cx + by + az)\times \\ \times\left(xU_x + yU_y + zU_z - mU\right) + \\ +mc\left((2m+1)xU - \left(x^2+y^2+z^2\right)U_x\right) + \\ +mb\left((2m+1)yU - \left(x^2+y^2+z^2\right)U_y\right) + \\ +ma\left((2m+1)xU - \left(x^2+y^2+z^2\right)U_z\right). \end{gathered} \tag{66}$$

It can be seen from this identity that operator (65) is in fact no different from a linear combination of operators (59), (60) and (61) while it is considered on a narrow subset consisting of homogeneous functions of degree $m$.

If we do not confine ourselves to first-order differential operators, we can equally use the three-dimensional Laplace equation as additives, instead of Euler relation (16),, as well as higher-order differential relations obtained by differentiating these differential equations with respect to the variables $x$, $y$ or $z$ multiplied by arbitrary functions. However, although such formulas can considerably differ (in the algebraic sense) from the formulas from the list obtained earlier, all these formulas are completely equivalent to basic formulas (21)–( 31) or their linear combinations with constant coefficients as generators of new analytical expressions for three-dimensional homogeneous harmonic functions. This relationship, however, is not always obvious (see, for example, expression (65)). Notably, Donkin operators are not an exception to this; in particular, similar problems on actual identity of mathematical expressions that are not identicall to each other in an algebraic sense are considered, for example, in [51–54], although with completely different data.

The problem on existence and determination of such Thomson formulas and such Donkin operators for homogeneous harmonic functions, with the arguments of the function $U$ replaced by variables, remains to be explored.

$$x \to f(x,y,z),\ y \to g(x,y,z),$$

$$z \to h(x,y,z).$$

Moreover, for a particular case of symmetrized substitution of variables

$$x \to x\varphi(x,y,z),\quad y \to y\varphi(x,y,z),$$

$$z \to z\varphi(x,y,z)$$

(see our studies [40, 41]) the common factor $\varphi(x,y,z)$ is taken out of the arguments of the homogeneous function $U$, and thus all such cases of substitution of variables coincide with the case already considered when the arguments of the function $U$ remain the same. However, study of additional cases for the Thomson formulas and Donkin operators that may appear when using substitution of general variables for the arguments of the function $U$ is beyond the scope of our investigation.

***Notes***

1. A general three-dimensional harmonic function $U(x, y, z)$ is completely determined by two arbitrary functions of two variables, for example, the value $U^{(0)}(x, y) = U(x, y, 0)$ of the function $U(x, y, z)$ along the plane $z = 0$ and the value

$$U^{(n)}(x,y) = \partial U(x,y,0)/\partial z$$

of the normal derivative function $U(x,y,z)$ along the plane $z = 0$.

This follows from the fact that the solution to such a Cauchy problem for the three-dimensional Laplace equation is uniquely determined (at least in the vicinity of the plane $z = 0$) using the Scherzer series with respect to the variable $z$:

$$\begin{aligned}U(x,y,z) &= U^{(0)}(x,y) + zU^{(n)}(x,y) - \\ &-\frac{z^2}{2!}\left(U^{(0)}_{xx}(x,y)+U^{(0)}_{yy}(x,y)\right) - \\ &-\frac{z^3}{3!}\left(U^{(n)}_{xx}(x,y)+U^{(n)}_{yy}(x,y)\right) + \qquad (67)\\ &+\frac{z^4}{4!}\left(U^{(0)}_{xxxx}(x,y)+2U^{(0)}_{xxyy}(x,y)+U^{(0)}_{yyyy}(x,y)\right) + \\ &+\frac{z^5}{5!}\left(U^{(n)}_{xxxx}(x,y)+2U^{(n)}_{xxyy}(x,y)+U^{(n)}_{yyyy}(x,y)\right) - \cdots,\end{aligned}$$

where the coefficients in front of the powers of the variable $z$ are expressed uniquely in terms of the functions $U^{(0)}$, $U^{(n)}$ and their partial derivatives, since the function $U$ should satisfy the Laplace equation.

A necessary and sufficient condition for obtaining a three-dimensional homogeneous harmonic function of degree $m$ is that the functions $U^{(0)}(x,y) = U(x,y,0)$ and

$$U^{(n)}(x,y) = \partial U(x,y,0)/\partial z$$

be Euler-homogeneous with degrees of homogeneity $m$ and $m - 1$ (sufficiency follows from formula (67) and the fact that the partial derivatives of homogeneous functions $U^{(0)}(x,y)$ and $U^{(n)}(x,y)$ are themselves homogeneous functions of the corresponding degree [17, 18]). However, an Euler-homogeneous function of two variables is defined using an arbitrary function of one variable, as follows from its representation in the form

$$f(x,y) = x^k g(y/x).$$

Thus, three-dimensional homogeneous functions are uniquely determined through two arbitrary functions of a single real variable, in contrast to three-dimensional harmonic functions of a general form.

2. While the process described in the introduction allows to calculate any homogeneous harmonic function with integer degrees of homogeneity, its drawback is that the same three-dimensional homogeneous harmonic function can be obtained multiple times from different two-dimensional harmonic prototypes. Donkin formulas (5), (6) are much more convenient in this regard, establishing a one-to-one correspondence between the three-dimensional homogeneous and harmonic functions $V_0$ and $V_{-1}$ and two-dimensional harmonic functions $H$. The procedure of homogeneous harmonic differentiation (more precisely, integration) with respect to the variables $x$, $y$, and $z$ does not have a similar useful property, since the prototype $U$ for a given function $V$ *is* reconstructed with some freedom of choice [44, 46].

3. It follows from Eq. (10) for the same spherical function $u(\theta,\varphi)$ for which the homogeneous function

$$V(x, y, z) = r^m u(\theta,\varphi)$$

is harmonic that the homogeneous function

$$V(x, y, z) = r^{-m-1} u(\theta, \varphi)$$

is also harmonic.

The reason for this is that the product $m(m + 1)$ in Eq. (10) for a spherical function $u(\theta, \varphi)$, actually preserving the function $V$ harmonic does not change with the substitution $m \to (-m - 1)$. This can be regarded as yet another proof of that Thomson transform (7) is harmonic for homogeneous functions, since

$$V(x, y, z) = r^{-m-1} u(\theta, \varphi) = r^{-2m-1} U(x, y, z).$$

The calculations in this paper were performed using the Wolfram Mathematica software [55].

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