DOI: 10.18721/JPM.12302
УДК 519.63:621.039

THE SYSTEM OF MULTIPHASE FLOW EQUATIONS IN THE EQUILIBRIUM BAROTROPIC APPROXIMATION: A NUMERICAL SCHEME

S.V. Bulovich

Peter the Great St. Petersburg Polytechnic University, St. Petersburg, Russian Federation

In the paper, an economical integration scheme of a differential equation system has been proposed for the model of multiphase medium with a common pressure in liquids. The algorithm allows someone to consider various cases of multiphase medium flows with formation of any structure by the number of liquids under study and admits an asymptotic transition to a single-fluid model. The algebraic balance relation of the fluid volume fraction was stated through the form of a differential equation in pressure. The correctness of the Cauchy problem for an equation system was remade using repeated derivatives of the sought-for functions with respect to a spatial coordinate.The Riemann problem in the varying-area channel at various values of liquid pressure and its volume fraction on the different sides of a diaphragm in the three-fluid version was solved.

Keywords: multi-fluid model, numerical simulation, correctness of Cauchy problem, economical algorithm

Citation: Bulovich S.V. The system of multiphase flow equations in the equilibrium barotropic approximation: a numerical scheme, St. Petersburg Polytechnical State University Journal. Physics and Mathematics. 12 (3) (2019) 17–24. DOI: 10.18721/JPM.12302

СХЕМА ЧИСЛЕННОГО ИНТЕГРИРОВАНИЯ СИСТЕМЫ УРАВНЕНИЙ МНОГОФАЗНЫХ ПОТОКОВ В РАВНОВЕСНОМ БАРОТРОПНОМ ПРИБЛИЖЕНИИ

С.В. Булович

Санкт-Петербургский политехнический университет Петра Великого,

Санкт-Петербург, Российская Федерация

Для модели многофазной среды с общим давлением в жидкостях предложена экономичная схема интегрирования системы дифференциальных уравнений. Алгоритм позволяет рассматривать варианты потока многофазной среды с формированием любой структуры по количеству рассматриваемых жидкостей и допускает асимптотический переход к одножидкостной модели. Алгебраическое соотношение баланса объемной доли жидкостей сформулировано в виде дифференциального уравнения относительно давления. Корректность задачи Коши для системы уравнений, описывающих поведение многофазной среды с общим давлением, восстанавливается путем использования повторных производных по пространственной координате от разыскиваемых функций. В качестве примера работы алгоритма рассмотрен вариант течения для трех жидкостей. Решена задача Римана о распаде разрыва в канале переменного сечения при различных значениях давления и объемной доли жидкостей по разные стороны диафрагмы.

Ключевые слова: многожидкостная модель, численное моделирование, корректность задачи Коши, экономичный алгоритм

Ссылка при цитировании: Булович С.В. Схема численного интегрирования системы уравнений многофазных потоков в равновесном баротропном приближении // Научно-технические ведомости СПбГПУ. Физико-математические науки. 2019. Т. 3 № .12. С. 17–24. DOI: 10.18721/JPM.12302

Introduction

A range of mathematical models have been formulated to describe the phenomena occurring in heterogeneous media; these models can be organized hierarchically depending on how complete they are and how satisfactorily they describe the given processes. The simplest models represent multiphase flow as a mixture described by a single equation for the balance of mass, momentum and energy. Rakhmatulin [1] introduced a fuller description, using the concept of interpenetrating continua, with a region of space characterized by the volume fractions of a substance in different aggregate states. The number of balance ratios used to determine the remaining thermal and gas-dynamic functions of the state of multiphase flow is chosen depending on the assumptions about the equilibrium of the processes. The concept of liquids can be involved if a broader interpretation of interpenetrating continua is adopted, assuming that the composition, level of velocities, temperatures, and pressures vary for the same state of aggregation. A model based on the assumption of equal pressures in liquids has gained wide recognition in this problem statement. These models are typically classified as pressure-balanced.

The mathematical model can be further simplified by considering barotropic processes in the class of flows where thermal processes do not play the governing role. In this case, the multi-fluid model consists of continuity equations, momentum equations, equations relating density and pressure for each of the liquids, and the algebraic balance ratio of the volume fraction of liquids. The system of equations written in a quasi-one-dimensional statement for a channel of variable cross-section has the form

$$\begin{gathered}\frac{\partial\left(A\alpha_k\rho_k\right)}{\partial t}+\frac{\partial\left(A\alpha_k\rho_k u_k\right)}{\partial x}=0,\\ \frac{\partial\left(A\alpha_k\rho_k u_k\right)}{\partial t}+\frac{\partial\left(A\alpha_k\rho_k u_k^2+A\alpha_k p\right)}{\partial x}-\\ -p\frac{\partial\left(A\alpha_k\right)}{\partial x}=0,\\ \sum_{k=1}^{k=K}\alpha_k=1,\ \rho_k=\rho_k\left(p\right),\end{gathered}\qquad(1)$$

where α_k is the volume fraction of the kth liquid; ρ_k, kg/m^3 and u_k, m/s, are its density and velocity ($k = 1, 2,..., K$; K is the total number of liquids); p, Pa, is the pressure in the channel cross-section that is the same for all liquids; A, m^2, is the area of the channel cross-section; t, s, is the time; x, m, is the coordinate along the generatrix of the channel.

System of differential equations (1) consists of terms in the form of first derivatives with respect to space and time. The source terms describing mass transfer in liquids and momentum change due to interaction of phases on the interface surface or to interaction of liquids with the channel wall are not included. It is uncommon to use differential relations to construct these terms and, therefore, such terms do not affect the type and properties of system of equations (1).

It was established within the framework of the two-fluid approach ($K = 2$) that system of equations (1) is always non-hyperbolic if the velocities in the liquids are not equal in magnitude and the Cauchy problem is incorrect for the system of equations. In other words, formally speaking, such a system of equations cannot describe the state of a heterogeneous medium [2]. Several techniques are known to solve this problem.

A formal approach to system of equations (1) as a system of first-order equations assumes that it has hyperbolic properties as a set of transfer equations, observed upon limit passage to single-phase flow (single liquid). The complex eigenvalues (treated as the propagation velocities of disturbances) in the Jacobi matrix for this system in case of two or more liquids indicate the absence of characteristic directions and loss of hyperbolic properties. Understanding this allows to develop an approach to regularizing the properties of the matrix, eliminating complex eigenvalues of the Jacobi matrix by adding certain terms containing the first derivatives of the desired functions.

Historically, added masses were taken into account in literature within this approach for reconstructing the hyperbolic properties of the system of Euler equations for multiphase problems in the formulation with total pressure. Adding the derivatives of the velocity components with respect to space and time to the momentum equations allows to determine the dimensions of the region without an unfavorable combination of the values of the desired functions [3–5]. Ultimately, by developing this direction, it proved sufficient to add only one term containing the volume fraction gradient as a factor for each liquids in the equation of momentum in system of equations (1). The physical interpretation of this term is associated with pressure at the boundary p_i (the index i here denotes the given interface, i.e., the pressure on

the surface between contacting liquids). In this case, the pressure difference $p_i - p$ can be found in terms of the characteristics of the contacting liquids. Selecting a quantitative value for the difference $p_i - p$ provides real values of all the eigenvalues of the Jacobi matrix. In particular, by analyzing the roots of the characteristic equation for two-fluid models we find that the real values are provided in selecting a quantitative difference as a quantity proportional to the liquid velocity relative to the interface velocity, i.e., $u_{ki} - u_k$. This technique was used for a two-fluid approach, for example, in [6–8]; the issue whether this technique is applicable to three or more liquids remains unexplored in literature.

We should mention one more technique for numerically solving system of equations (1). It consists in using difference operators approximating derivatives of the first-order accuracy to represent derivatives with respect to space. In this case, the first differential approximation of the difference scheme contains the iterative derivative as the main term, which changes the type of system of differential equations from indefinite to parabolic. The influence of these derivatives in the system of equations of the first differential approximation depends on the coefficients of the highest derivatives, i.e., on the scheme viscosity. It was found that a numerical solution to system of equations (1) can be obtained with coarse discretization of the computational domain, i.e., with high values of the scheme viscosity. Unfortunately, it is impossible to establish the asymptotic behavior of the solution with decreasing grid spacing, since the oscillating solutions are exponential with decreasing scheme viscosity.

There are two important considerations to be borne in mind in solving system of equations (1) in this study.

Firstly, the type of the given system of equations was changed to parabolic by supplementing the momentum equations with iterative derivatives. This way, the stability of the computational process is not generally associated with approximating operators with first-order accuracy representing the derivatives with respect to space. The necessary transfer coefficient (similar to the interpretations adopted in computational mathematics, it should be called artificial viscosity) is determined by the parameters of the problem solved: the properties of the given liquids and the spatial discretization step.

Secondly, an evolutionary differential equation of the parabolic type with respect to pressure was formulated instead of the algebraic balance equation of the volume fraction.

To formulate the given equation, let us use the continuity equation. Writing the value of the derivative of the product of functions with respect to time for each liquid, we obtain an equation of the following form:

$$A\frac{\partial \alpha_k}{\partial t} + A\frac{\alpha_k}{\rho_k}\frac{\partial \rho_k}{\partial t} + \frac{1}{\rho_k}\frac{\partial\left(A\alpha_k\rho_k u_k\right)}{\partial x} = 0. \quad (2)$$

Using the equation of state for each of the liquids, replacing the derivative of density with the derivative of pressure and summing over all volume fractions taking into account the algebraic balance ratio of the volume fraction, we can obtain the evolution equation for pressure, which has a clear physical meaning as the resultant measure of imbalance of mass flows of different liquids.

$$A\left(\sum_{k=1}^{K}\frac{\alpha_k}{\rho_k c_k^2}\right)\frac{\partial p}{\partial t} + \sum_{k=1}^{K}\left(\frac{1}{\rho_k}\frac{\partial\left(A\alpha_k\rho_k u_k\right)}{\partial x}\right) = 0. \quad (3)$$

The factor in the form of the sum before the derivative of pressure with respect to time in the first term actually serves for calculating the speed of sound in a mixture of liquids by the Wood formula.

After these two modifications, system of equations (1) takes the following form:

$$\begin{gathered}
\frac{\partial\left(A\alpha_k\rho_k\right)}{\partial t} + \frac{\partial\left(A\alpha_k\rho_k u_k\right)}{\partial x} = 0, \\
\frac{\partial\left(A\alpha_k\rho_k u_k\right)}{\partial t} + \frac{\partial\left(A\alpha_k\rho_k u_k^2\right)}{\partial x} + A\alpha_k\frac{\partial p}{\partial x} = \\
= \frac{\partial}{\partial x}\left(\nu A\frac{\partial\left(\alpha_k\rho_k u_k\right)}{\partial x}\right), \\
\frac{A}{\rho c^2}\frac{\partial p}{\partial t} + \sum_{k=1}^{K}\left(\frac{1}{\rho_k}\frac{\partial\left(A\alpha_k\rho_k u_k\right)}{\partial x}\right) = 0, \\
\rho_k = \rho_k\left(p\right),\ c_k^2 = \frac{\partial p}{\partial \rho_k},\ \rho = \sum_{k=1}^{K}\alpha_k\rho_k, \\
\frac{1}{\rho c^2} = \sum_{k=1}^{K}\left(\frac{\alpha_k}{\rho_k c_k^2}\right),\ \nu = \Delta x c,
\end{gathered} \quad (4)$$

where c_k and c, m/s, are, respectively the speed of sound in the kth liquid and that in a heterogeneous medium, found by the Wood formula; ρ is the density of the mixture.

The artificial viscosity coefficient was chosen based on the study conducted in [9].

Numerical method

Let us use the finite difference method to numerically solve system of differential equations (4). For this purpose, let us introduce a uniform spatial grid with the discretization step Δx and the integration time step Δt; f_i^n is the function value given in the grid node with the coordinates $i\Delta x$ and $n\Delta t$. Let us use the simplest difference relations to describe the algorithm:

$$\Lambda f_i = \left(f_i - f_{i-1}\right)/\Delta x$$

is the two-point directed difference operator approximating the derivative $\partial/\partial x$ with first-order accuracy;

$$\overline{\Lambda} f_i = \left(f_{i+1} - f_i\right)/\Delta x$$

is the directed difference operator conjugate to Λ.

The simplest algorithm for numerical integration of system of equations (4) is based on the explicit-implicit approximation of its terms.

Finite-difference approximation for K momentum equations for each of the liquids with $(u_k)_i^n > 0$ has the form:

$$\frac{\left(A\alpha_k\rho_k u_k\right)_i^{n+1} - \left(A\alpha_k\rho_k u_k\right)_i^n}{\Delta t} + \Lambda\left(A\alpha_k\rho_k u_k^2\right)_i^n - \Lambda\left(A\nu\overline{\Lambda}\left(\alpha_k\rho_k u_k\right)_i^n\right) + A\left(\alpha_k\right)_i^n \overline{\Lambda} p_i^{n+1} = 0. \quad (5)$$

The finite-difference approximation of the evolution equation with respect to pressure can be written in the form

$$\left(\frac{A}{\rho c^2}\right)_i^n \frac{p_i^{n+1} - p_i^n}{\Delta t} + \sum_{k=1}^{K}\left(\frac{1}{\left(\rho_k\right)_i^n}\Lambda\left(A\alpha_k\rho_k u_k\right)_i^{n+1}\right) = 0. \quad (6)$$

An important point is that an operator conjugate to the convective transfer operator is used to approximate the pressure gradient. This leads, in case of multidirectional motion of liquids, to coupling of the pressure of the current point with two neighboring ones. Notably, if $(u_k)_i^n < 0$, cyclic substitution of Λ to $\overline{\Lambda}$ is done in formulas (5) and (6).

The set (K equations (5) and equation (6)) forms a closed system, where the variable $(A\alpha_k\rho_k u_k)_i^{n+1}$ can be excluded by direct substitution, and the equation with respect to pressure p_i^{n+1} can be formulated. If we introduce the notation for a group of terms defined on a known time layer n by the formula

$$\left(A\alpha_k\rho_k u_k\right)_i^* = \left(A\alpha_k\rho_k u_k\right)_i^n - \Delta t\left(\Lambda\left(A\alpha_k\rho_k u_k^2\right)_i^n - \Lambda\left(A\nu\overline{\Lambda}\left(\alpha_k\rho_k u_k\right)_i^n\right)\right), \quad (7)$$

then the pressure in the channel is found from the solution of an equation of the form

$$\left(\frac{A}{\rho c^2}\right)_i^n p_i^{n+1} - \Delta t^2\sum_{k=1}^{K}\left(\frac{1}{\left(\rho_k\right)_i^n}\Lambda\left(A\alpha_k\overline{\Lambda}p_i^{n+1}\right)\right) = -\Delta t\sum_{k=1}^{K}\left(\frac{1}{\left(\rho_k\right)_i^n}\Lambda\left(A\alpha_k\rho_k u_k\right)_i^*\right) + \left(\frac{A}{\rho c^2}\right)_i^n p_i^n. \quad (8)$$

Notice that any sequence combinations of operators

$$\Lambda(A\alpha_k\overline{\Lambda}p_i^{n+1}) \text{ or } \overline{\Lambda}(A\alpha_k\Lambda p_i^{n+1}),$$

which are essentially operators representing the iterative derivative, are equivalent to each other. The standard representation of the iterative derivative generates a tridiagonal matrix of coefficients in front of unknown values p_i^{n+1}. To obtain a unique solution to the problem, we should formulate two boundary conditions with respect to pressure at different ends of the channel.

Let us consider the formulation of boundary conditions for a permeable and for an impermeable boundary. The following values are typically given for an inflow boundary, when all liquids enter the computational domain:

$$\alpha_k^{inlet}, \rho_k^{inlet}, u_k^{inlet},$$

i.e., the flow rate is given for each liquid

$$G_k^{inlet} = A\alpha_k^{inlet}\rho_k^{inlet}u_k^{inlet}.$$

Assuming that variation in these values over time does not produce significant pressure gradients in the vicinity of the boundary, we can use a homogeneous boundary condition of the second kind

$$\frac{\partial p}{\partial x} = 0, \quad (9)$$

which can be represented as a two-point relation in a difference form. The same boundary condition can be applied to a blind channel (impermeable

end surface of the channel), which is immediate from the equation of momentum at zero flow rate.

The pressure level at the outlet boundary, when all liquids leave the computational domain, can be found by a boundary condition of the first kind:

$$p = p^{outlet}. \qquad (10)$$

The values of the remaining functions at the outlet boundary for the liquids leaving the computational domain are determined assuming their smooth behavior, i.e., by extrapolation.

The case with a permeable boundary when a number of liquids enter the computational domain with the remaining liquids simultaneously leaving it is not considered.

After finding the pressure from equation (8) using the standard scalar sweep procedure with two-point boundary conditions, the remaining values of the functions are calculated by explicit formulas for all K liquids in the following sequence:

$$\begin{aligned} \left(A\alpha_k\rho_k u_k\right)_i^{n+1} &= \left(A\alpha_k\rho_k u_k\right)_i^{*} - \\ &-\Delta t\alpha_k A\bar{\Lambda}p_i^{n+1}, \\ \left(A\alpha_k\rho_k\right)_i^{n+1} &= \left(A\alpha_k\rho_k\right)_i^{n} - \\ &-\Delta t\Lambda\left(A\alpha_k\rho_k u_k\right)_i^{n+1}. \end{aligned} \qquad (11)$$

The scheme for discretization of variables described in the paper and the difference operators applied yield a first-order approximation for the system of differential equations with respect to time and space. The time limit for the integration step, associated with the explicit form for describing convective and diffusive flows, is found by standard estimates:

$$\Delta t \le \min\left(\frac{\Delta x}{c}, \quad \frac{\Delta x^2}{2\nu}\right). \qquad (12)$$

Judging from inequality (12), the time limit imposed on the integration step for the given method of determining the artificial viscosity coefficient is twice as stringent for the explicit approximation of the diffusion terms compared to the condition resulting from the estimate of the stability of the computational process associated with convective transfer.

Notably, artificial "parabolization" of system of equations (1) with respect to all variables opens up wider possibilities than reconstruction of hyperbolic properties of system of equations (1). In particular, more variable functional values are admissible for the equation of state. For example, multiphase flows with incompressible inclusions can be considered, with the equation of state reduced to a density constant (in this case, the speed of sound in this component is equal to infinity).Passage to the limit where all components of multiphase flow are incompressible can be made. In this boundary case, an unsteady term is absent in equation (6) (the type of equation for pressure changes to elliptic) and an asymptotic transition to the classical model of incompressible liquid, generalized to several components, occurs.

Example calculation

To illustrate the proposed algorithm, we have carried out calculations of the flow developing in a channel of variable cross-section connecting two reservoirs in the three-fluid approximation. The problem statement

Table

Quantitative values of thermodynamic functions and their equations of state for three liquids

Thermodynamic function	Value of function or its equation		
	Vapor	Incompressible liquid	Weakly compressible liquid
Pressure, MPa in the left tank in the right tank	 2.0 0.2	 2.0 0.2	 2.0 0.2
Volume fraction in the left tank in the right tank	 0.40 0.59	 0.59 0.40	 0.01 0.01
Material density, kg/m^3	$\left(\frac{p}{1.0e5}\right)^{0.714}$	1.0e3	$1.0e3\left(\frac{p}{1.0e6}\right)^{0.130}$

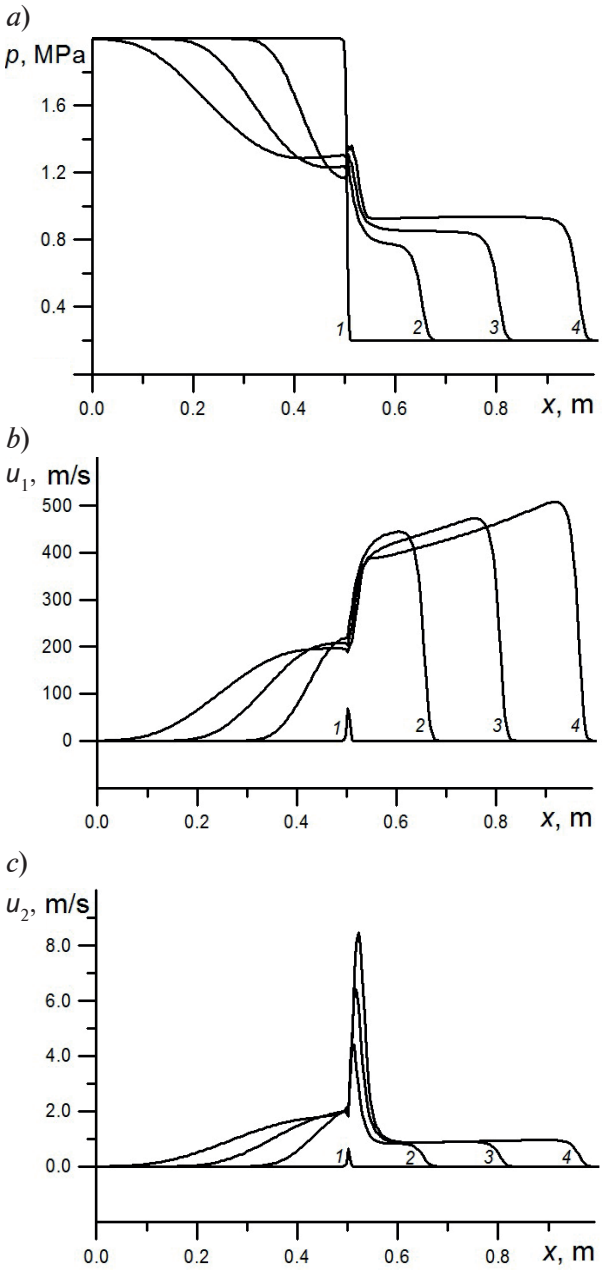

Fig. 1. Calculated profiles of gas-dynamic pressure (*a*), velocities of vapor (*b*) and incompressible liquid (*c*) along the longitudinal coordinate at first (*1*), 100th (*2*), 200th (*3*) and 300th (*4*) time steps

was taken from [10], where the solution was obtained by the procedure implemented in the KORSAR code [11] in the three-fluid approximation. The solution and the graphs given show the behavior of the functions appearing upon instantaneous fracture of the aperture located in the middle of the channel.

The quantitative values of thermodynamic functions used and their equations of state for liquids are summarized in Table.

Unsteady flow developing from a state of rest in a conical channel with an area ratio of 4:1 and a length of 1 m composed of three liquids (see Table) was simulated on a mesh consisting of 256 nodes. The integration time step chosen was 2.0e−6 s.

The result of numerical integration after 300 time steps is shown in Fig. 1, giving the profiles of gas-dynamic functions of the first and each 100th time step. In particular, the figure shows the pressure (*a*) and velocity of vapor (*b*) and incompressible liquid (*c*) as functions of the longitudinal coordinate at selected points in time.

Conclusion

Let us now summarize the key advantages of the algorithm proposed.

Firstly, the algorithm does not place a limit on the number of liquids.

Secondly, the characteristics of the flow in the initial stages of fast processes can be obtained by regularizing the Cauchy problem using iterated derivatives without distorting the propagation velocity of acoustic disturbances.

Thirdly, the algorithm is more effective than other existing algorithms for solving systems of equations for the two-fluid (multifluid) approach, since the calculations are carried out by explicit formulas and by one scalar sweep.

The study was carried out as part of State Task of the Ministry of Education and Science of the Russian Federation No. 3.3314.2017/4.6 (2017–2019).

REFERENCES

1. **Rakhmatulin Kh.A.,** Osnovy gazodinamiki vzaimopronikayushchikh dvizheniy szhimayemykh sred [Gas dynamics principles of interpenetrating motions of compressible media] J. Appl. Math. Mech. 20 (2) (1956) 184–195.

2. **Klebanov L.A., Kroshilin A.E., Nigmatulin B.I., Nigmatulin R.I.,** O giperbolichnosti, ustoychivosti i korrektnosti zadachi Koshi dlya sistemy differentsialnykh uravneniy dvukhskorostnogo dvizheniya dvukhfaznykh sred [On hyperbolicity, stability and correctness of the Cauchy problem for a differential equations system of a two-speed motion of biphasic media], J. Appl. Math. Mech. 46 (1) (1982) 83–95.

3. **Lyczkowski R.W., Gidaspow D., Solbrig Ch.W., Hughes E.D.,** Characteristics and

stability analysis of transient one-dimensional two-phase equiations and their finite-difference approximation, Nuclear Science and Engineering. 66 (3) (1978) 378–396.

4. **Chao B.T., Sha W.T., Soo S.L.,** On inertial coupling in dynamic equations of components in a mixture, International Journal of Multiphase Flow. 4 (2) (1978) 219–222.

5. **Drew D.A., Lahey R.T.,** Application of general constitutive principles to the derivation of multidimensional two-phase flow equations, International Journal of Multiphase Flow. 5 (4) (1979) 243–264.

6. **Figueiredo A.B., Baptista R.M., Rachid F.B.F., Bodstein G.C.R.,** Numerical simulation of stratified-pattern two-phase flow in gas pipelines using a two-fluid model, International Journal of Multiphase Flow. 88 (January) (2017) 30–49.

7. **Evje S., Flatten T.,** On the wave structure of two-phase flow models, SIAM J. Appl. Math. 67 (2) (2007) 487–511.

8. **Kroshilin A.E., Kroshilin V.E.,** Correct numerical simulation of a two-phase coolant, Thermal Engineering. 63 (2) (2016) 98–106.

9. **Bulovich S.V., Smirnov E.M.,** Experience in using a numerical scheme with artificial viscosity at solving the Riemann problem for a multi-fluid model of multiphase flow, AIP Conference Proceedings. 1959 (1) (2018) 050007.

10. **Smirnov S.I., Bulovich S.V., Smirnov E.M.,** Calculation of unsteady two-phase quasi-onedimensional channel flow based on the two-fluid model and the artificial viscosity numerical scheme, Proceedings of International Conference PhysicA, SPb/2018. St. Petersburg, October, 23–25, (2018), Journal of Physics: Conference Series. 1135. 012103. 10.1088/1742-6596/1135/1/012103.

11. **Migrov Yu.A., Volkova S.N., Yudov Yu.V., et al.,** KORSAR –teplogidravlicheskiy raschetnyy kod novogo pokoleniya dlya obosnovaniya bezopasnosti AES s VVER [KORSAR is a heat-hydraulic new-generation design code for a safety case of a NPP with LWR], Thermal Engineering. (9) (2001) 36–43.

Received 26.03.2019, accepted 13.05.2019.

THE AUTHOR

BULOVICH Sergei V.
Peter the Great St. Petersburg Polytechnic University
29 Politechnicheskaya St., St. Petersburg, 195251, Russian Federation
bulovic@yandex.ru

СПИСОК ЛИТЕРАТУРЫ

1. **Рахматулин Х.А.** Основы газодинамики взаимопроникающих движений сжимаемых сред // Прикладная математика и механика. 1956. Т. 2 № .20. С. 184–195.

2. **КлебановЛ.А., КрошилинА.Е., Нигматулин Б.И., Нигматулин Р.И.** О гиперболичности, устойчивости и корректности задачи Коши для системы дифференциальных уравнений двухскоростного движения двухфазных сред // Прикладная математика и механика. 1982. Т. 46. № 1. С. 83–95.

3. **Lyczkowski R.W., Gidaspow D., Solbrig Ch.W., Hughes E.D.** Characteristics and stability analysis of transient one-dimensional two-phase equiations and their finite-difference approximation // Nuclear Science and Engineering. 1978. Vol. 66. No. 3. Pp. 378–396.

4. **Chao B.T., Sha W.T., Soo S.L.** On inertial coupling in dynamic equations of components in a mixture// International Journal of Multiphase Flow. 1978. Vol. 4. No. 2. Pp. 219–222.

5. **Drew D.A., Lahey R.T.** Application of general constitutive principles to the derivation of multidimensional two-phase flow equations // International Journal of Multiphase Flow. 1979. Vol. 5. No. 4. Pp. 243–264.

6. **Figueiredo A.B., Baptista R.M., Rachid F.B.F., Bodstein G.C.R.** Numerical simulation of stratified-pattern two-phase flow in gas pipelines using a two-fluid model // International Journal of Multiphase Flow. 2017. Vol. 88. January. Pp. 30–49.

7. **Evje S., Flatten T.** On the wave structure of two-phase flow models // SIAM J. Appl. Math. 2007. Vol. 67. No. 2. Pp. 487–511.

8. **Крошилин А.Е., Крошилин В.Е.** Корректное численное моделирование двухфазного теплоносителя // Теплоэнергетика. 2016. № 2. С. 22–30.

9. **Bulovich S.V., Smirnov E.M.** Experience in using a numerical scheme with artificial vis-

cosity at solving the Riemann problem for a multi-fluid model of multiphase flow // AIP Conference Proceedings. 2018. Vol. 1959. No. 1. P. 050007.

10. **Smirnov S.I., Bulovich S.V., Smirnov E.M.** Calculation of unsteady two-phase quasi-onedimensional channel flow based on the two-fluid model and the artificial viscosity numerical scheme // Proceedings of International Conference PhysicA. SPb/2018. St. Petersburg, October, 23–25, 2018. Journal of Physics: Conference Series. 1135. 012103. 10.1088/1742-6596/1135/1/012103.

11. **Мигров Ю.А., Волкова С.Н., Юдов Ю.В. и др.** КОРСАР – теплогидравлический расчетный код нового поколения для обоснования безопасности АЭС с ВВЭР // Теплоэнергетика. 9 № .2001. С. 36–43.

Статья поступила в редакцию 26.03.2019, принята к публикации 13.05.2019.

СВЕДЕНИЯ ОБ АВТОРЕ

БУЛОВИЧ Сергей Валерьевич – *кандидат физико-математических наук, доцент кафедры «Гидроаэродинамика, горение и теплообмен» Санкт-Петербургского политехнического университета Петра Великого.*

195251, Российская Федерация, г. Санкт-Петербург, Политехническая ул., 29

bulovic@yandex.ru