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GENERALIZATION OF THE THOMSON FORMULA FOR HOMOGENEOUS HARMONIC FUNCTIONS

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In the paper, it has been shown that the Thomson formula for three-dimensional harmonic homogeneous functions in Euler terms can be generalized using a linear algebraic form involving the first order partial derivatives of the initial function instead of pure algebraic linear expressions. An exhaustive list of the formed first order expressions converting arbitrary three-dimensional harmonic functions in Euler terms into new three-dimensional homogeneous harmonic functions was presented.

Keywords: electrostatic field, magnetostatic field, scalar potential, homogeneous function, harmonic function

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ОБОБЩЕНИЕ ФОРМУЛЫ ТОМСОНА ДЛЯ ГАРМОНИЧЕСКИХ ОДНОРОДНЫХ ФУНКЦИЙ

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В работе показано, что формулу Томсона для трехмерных гармонических функций, однородных по Эйлеру, можно обобщить, если вместо чисто алгебраических линейных выражений использовать линейную алгебраическую форму с участием частных производных первого порядка от исходной функции. Приводится исчерпывающий список получающихся выражений первого порядка, преобразующих произвольные трехмерные однородные гармонические функции в новые трехмерные гармонические функции.

Ключевые слова: Ключевые слова: электростатическое поле, магнитостатическое поле, скалярный потенциал, уравнение Лапласа, формула Томсона

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Introduction

This paper continues the studies in [1], considering generalizations of the Thomson formula for 3D harmonic functions of a general form. Differential-algebraic transformations similar to the classical Thomson formula for homogeneous harmonic functions were discussed. These transformations can be used to generate new harmonic functions represented in analytical form that are Euler-homogeneous. Electric and magnetic fields whose scalar potential (3D harmonic function) is an Euler-homogeneous function obey the principle of similarity of trajectories described by Golikov [2, 3]. Such fields possess additional useful electron and ion-= optical properties [4—10].

Let $r=\sqrt{x^2+y^2+z^2}$ be the distance from a sample point (x,y,z) to the origin. The Thomson formula (Kelvin transform)

$$V(x,y,z)=\frac{1}{r}U\left(\frac{x}{r^2},\frac{y}{r^2},\frac{z}{r^2}\right). \quad (1)$$

transforms the harmonic function $U(x,y,z)$, satisfying the Laplace equation

$$U_{xx} + U_{yy} + U_{zz} = 0, \quad (2)$$

into a new harmonic function $V(x,y,z)$ [11—18]. From now on we use the subscripts x, y, z to denote partial derivatives with respect to the corresponding variables.

In particular, the Thomson formula (1) serves as a useful mathematical tool for generating analytical expressions for scalar potentials of electric and magnetic fields that can be used for synthesizing new electron and ion optical systems [4—10, 19, 20].

The function $U(x,y,z)$ is called Euler-homogeneous (or, more precisely, positively homogeneous) if it satisfies the identity

$$\forall\lambda>0:\ U(\lambda x,\lambda y,\lambda z)\equiv\lambda^m U(x,y,z),$$

where m is the degree of homogeneity of the function (not necessarily an integer) [21, 22].

For a continuously differentiable function $U(x,y,z)$ to be positively homogeneous with degree m, it is necessary and sufficient that the Euler differential equation for homogeneous functions be satisfied at any point in space (except for the origin):

$$xU_x + yU_u + zU_z - mU = 0. \quad (3)$$

The proof of this important statement can be found in [21, 22].

Searching among linear algebraic formulas of the type

$$V(x,y,z)=S(x,y,z)\times \times U\big(f(x,y,z),g(x,y,z),h(x,y,z)\big) \quad (4)$$

(where S, f, g, h are some fixed functions) of other relations that can generate new harmonic functions $V(x,y,z)$ for any harmonic functions $U(x,y,z)$, we verified that the Thomson formula (1) is, in a sense, unique [1, 15, 17].

Specifically, the Thomson formula (1) and the trivial identity $V(x,y,z) = U(x,y,z)$ are the only expressions of form (4) that meet the requirements of the problem stated to within 3D rotations around the origin, shifts (parallel translations)

$$x' = x + a,\ y' = y + b,\ z' = z + c,$$

symmetries

$$x' = -x,\ y' = -y,\ z' = -z$$

and proportional scaling

$$x' = kx,\ y' = ky,\ z' = kz$$

(as well as multiplication of the potentials obtained by a constant factor).

Considering linear expressions including first partial derivatives of the function U, which have the form

$$\begin{aligned}V(x,y,z)=&\\ =S(x,y,z)\cdot U\big(f(x,y,z),g(x,y,z),h(x,y,z)\big)&+\\ +P(x,y,z)\cdot U_x\big(f(x,y,z),g(x,y,z),h(x,y,z)\big)&+\\ +Q(x,y,z)\cdot U_y\big(f(x,y,z),g(x,y,z),h(x,y,z)\big)&+\\ +R(x,y,z)\cdot U_z\big(f(x,y,z),g(x,y,z),h(x,y,z)\big)&\end{aligned} \quad (5)$$

(S, P, Q, R, f, g, h are some fixed functions), we found that additional formulas transforming 3D harmonic functions into new 3D harmonic functions could exist [1].

Now the transformations are not purely algebraic but differential-algebraic. This expands the range of mathematical tools available for generating new 3D harmonic functions that serve as scalar potentials for electric and magnetic fields.

Increasing the order of partial derivatives of the function U included in expressions of form (5) expands the list of such formulas even further but also makes operations with them somewhat inconvenient. In particular, such formulas cannot be uniquely defined because the second derivatives of harmonic functions depend on each other. This is to say that there is a wide variety of such formulas for each transformation: while their forms are not algebraically equivalent to each other, they yield essentially the same results.

Notably, however, we found a smaller amount of transformations preserving 3D harmonic functions in [1] than we had expected. One of the reasons why the search for new first-order algebraic and differential-algebraic expressions for generating 3D harmonic functions yielded unsatisfactory results might be that the requirement for these expressions to work for any initial harmonic functions $U(x,y,z)$ is too strict. For example, it is well known that any conformal transformation of the arguments for two-dimensional harmonic functions $U(x,y)$ generates a new two-dimensional harmonic function [23–27]. Such a transformation of harmonic functions has the form (4) but it is considerably different from the Thomson formula (1).

Another example is the Thomson formula for homogeneous harmonic functions (see [28] and [29], Appendix B to Chapter 1), which has the form

$$V(x,y,z) = r^{-2m-1}U(x,y,z). \qquad (6)$$

Function (6) becomes harmonic if a homogeneous harmonic function U of degree m substituted into it, but this is definitely not the case for any harmonic function U.

Finally, let us consider transformations of the form

$$\begin{gathered} V(x,y,z) = CU_x(x,y,z) + \\ +BU_y(x,y,z) + AU_z(x,y,z); \\ A = a(2m+1)xz + b(2m+1)yz + \\ +c\left(-mx^2 - my^2 + (m+1)z^2\right); \\ B = a(2m+1)xy + \\ +b\left(-mx^2 + (m+1)y^2 - mz^2\right) + \\ +c(2m+1)yz; \\ C = a\left((m+1)x^2 - my^2 - mz^2\right) + \\ +b(2m+1)xy + c(2m+1)xz, \end{gathered} \qquad (7)$$

that can be obtained by analyzing the harmonic conditions for linear forms of partial derivatives U_x, U_y, U_z with coefficients that are general quadratic forms in terms of variables x, y, z. It follows from the identity

$$\begin{gathered} V_{xx} + V_{yy} + V_{zz} \equiv \\ \equiv (4m+2)(ax+by+cz)\left(U_{xx} + U_{yy} + U_{zz}\right) + \\ +C\left(U_{xxx} + U_{xyy} + U_{xzz}\right) + B\left(U_{xxy} + U_{yyy} + U_{yzz}\right) + \\ +A\left(U_{xxz} + U_{yyz} + U_{zzz}\right) + \\ +2a\left(xU_{xx} + yU_{xy} + zU_{xz} - (m-1)U_x\right) + \\ +2b\left(xU_{xy} + yU_{yy} + zU_{yz} - (m-1)U_y\right) + \\ +2c\left(xU_{xz} + yU_{yz} + zU_{zz} - (m-1)U_z\right) \end{gathered}$$

that, generally speaking, if the only requirement imposed on the function U is for it to be harmonic, expression (7) is a harmonic function only if

$$a = b = c = 0.$$

If U is a homogeneous function of degree m, satisfying both the Laplace equation (2) and the Euler differential equation (3), then the conditions

$$\begin{gathered} U_{xxx} + U_{xyy} + U_{xzz} = 0, \\ U_{xxy} + U_{yyy} + U_{yzz} = 0, \\ U_{xxz} + U_{yyz} + U_{zzz} = 0, \\ xU_{xx} + yU_{xy} + zU_{xz} = (m-1)U_x, \\ xU_{xy} + yU_{yy} + zU_{yz} = (m-1)U_y, \\ xU_{xz} + yU_{yz} + zU_{zz} = (m-1)U_z, \end{gathered}$$

obtained by differentiating (2) and (3) with respect to x, y, z, are satisfied.

Then expression (7) turns out to be a harmonic homogeneous function of degree $(m+1)$ for any a, b, c.

The goal of this study has consisted in searching for alternative expressions that could be applicable for 3D homogeneous harmonic functions and useful for generating analytical expressions for 3D harmonic functions that are Euler-homogeneous.

There is good reason to believe that the set of transformations can be significantly expanded by limiting the class of transformed harmonic functions.

Problem statement

Let us consider the transformation of a 3D homogeneous harmonic function $U(x,y,z)$ in accordance with rule (5), where S, P, Q, R, f, g, h are some fixed functions, and U is an arbitrary homogeneous harmonic function of a fixed degree m. We require that expression (5) be an Euler-homogeneous function for the given functions S, P, Q, R, f, g, h and any homogeneous harmonic functions U of degree m.

Since the function U should satisfy the Euler differential equation (3), we can assume for expression (5) without loss of generality that

$$R(x, y, z) = 0$$

(the partial derivative of U_z can be expressed in terms of the functions U, U_x, U_y).

Furthermore, let us confine ourselves to a particular case

$$f(x,y,z) \equiv x,\ g(x,y,z) \equiv y,$$
$$h(x,y,z) \equiv z,$$

which is a consequence of the symmetric change of variables

$$f(x,y,z) = x\varphi(x,y,z),$$
$$g(x,y,z) = y\varphi(x,y,z),$$
$$h(x,y,z) = z\varphi(x,y,z)$$

considered in [1] (here the common factor $\varphi(x,y,z)$ is taken out of the arguments of the functions U, U_x, U_y and U_z because these functions are Euler-homogeneous).

Thus, we are going to consider transformations of the following type below:

$$\begin{aligned} V(x,y,z) &= S_0(x,y,z)\cdot U(x,y,z)+\\ &+P_0(x,y,z)\cdot U_x(x,y,z)+\\ &+Q_0(x,y,z)\cdot U_y(x,y,z), \end{aligned} \tag{8}$$

where S_0, P_0, Q_0 are some fixed functions preserving the function V homogeneous and harmonic provided that the function U with a given degree of homogeneity m is homogeneous and harmonic.

For expression (5) to be an Euler-homogeneous function when an arbitrary homogeneous harmonic function U is substituted into it, the functions S, P, Q, R, f, g, h in expression (5) should satisfy some additional conditions.

In particular, we can prove for expressions of form (8) that the condition that S_0 is Euler-homogeneous of degree $n - m$ and P_0 and Q_0 are Euler-homogeneous of degree $n - m + 1$, is sufficient as well as necessary for expression (8) to be a homogeneous function of degree n.

To prove that it is necessary for the functions S_0, P_0 and Q_0 to be Euler-homogeneous, we use the condition that the function U is Euler-homogeneous without invoking the condition that it is harmonic. We should also verify that there are no additional options for homogeneous harmonic functions that are missing in case of arbitrary homogeneous functions. However, the constraints

$$f(x,y,z) = x,\ g(x,y,z) = y,$$
$$h(x,y,z) = z,\ R(x,y,z) = 0$$

guarantee that there are no other possible options for the functions S_0, P_0 and Q_0.

The obvious method for solving the problem is to remove all dependent partial derivatives of the function U from the result of substituting the expression (5) or (8) into the Laplace equation and into the Euler differential equation. After that, the factors grouped before the remaining partial derivatives of the function U should be equal to zero separately and independently of each other. “Extra” derivatives should be excluded, using not only the Laplace equation

$$U_{xx} + U_{yy} + U_{zz} = 0,$$

as done in [1], but also the Euler differential equation

$$xU_x + yU_y + zU_z = mU$$

for homogeneous functions of degree m, as well as the results of differentiation of the given equations with respect to the variables x, y, z.

The task is somewhat complicated by the fact that differential relations dependent on each other appear upon independent differentiation of the Laplace equation and the Euler differential relation:

$$\begin{aligned} &\left(xU_{xxx}+yU_{xxy}+zU_{xxz}-(m-2)U_{xx}\right)+\\ &+\left(xU_{xyy}+yU_{yyy}+zU_{yyz}-(m-2)U_{yy}\right)+\\ &+\left(xU_{xzz}+yU_{yzz}+zU_{zzz}-(m-2)U_{zz}\right)\equiv\\ &\equiv x\left(U_{xxx}+U_{xyy}+U_{xzz}\right)+\\ &+y\left(U_{xxy}+U_{yyy}+U_{yzz}\right)+\\ &+z\left(U_{xxz}+U_{yyz}+U_{zzz}\right)-\\ &-(m-2)\left(U_{xx}+U_{yy}+U_{zz}\right). \end{aligned}$$

However, there is a more effective way to solve the problem. Since $U(x,y,z)$ is an Euler-homogeneous function of degree m, by applying the change of variables used for the Donkin formula [7–10, 30–34], we can formulate it as

$$U(x,y,z) = r^m F\left(\frac{x}{z+r}, \frac{y}{z+r}\right), \tag{9}$$

where $r=\sqrt{x^2+y^2+z^2}$ and F is some appropriate function of two variables.

This formulation is a slightly modified universal representation of homogeneous functions of degree k in accordance with the formula [21, 22]:

$$f\left(x_1,x_2,\ldots,x_n\right)=x_1^k g\left(x_2/x_1,\ldots x_n/x_1\right);$$

it does not lead to loss of admissible solutions.

Accordingly, the transformed function $V(x,y,z)$, which should be an Euler-homogeneous function of degree n, can be represented as

$$V\left(x,y,z\right)=r^n G\left(\frac{x}{z+r},\frac{y}{z+r}\right). \qquad (10)$$

The change of variables used to construct substitutions (9) and (10) is reversible:

$$\begin{cases} p=\dfrac{x}{z+\sqrt{x^2+y^2+z^2}}, \\ q=\dfrac{y}{z+\sqrt{x^2+y^2+z^2}}, \\ r=\sqrt{x^2+y^2+z^2}, \end{cases} \Leftrightarrow \qquad (11)$$

$$\Leftrightarrow \begin{cases} x=\dfrac{2pr}{1+p^2+q^2}, \\ y=\dfrac{2qr}{1+p^2+q^2}, \\ z=\pm\dfrac{r\left(1-p^2-q^2\right)}{1+p^2+q^2}. \end{cases}$$

For the function U, given by equality (9), to be harmonic, that is, to satisfy the 3D Laplace equation, it is necessary and sufficient that the function F satisfy the two-dimensional elliptic equation

$$\frac{\partial^2 F\left(p,q\right)}{\partial p^2}+\frac{\partial^2 F\left(p,q\right)}{\partial q^2}+ +\frac{4m\left(m+1\right)}{\left(1+p^2+q^2\right)^2}F\left(p,q\right)=0. \qquad (12)$$

Then any homogeneous harmonic function U of degree m corresponds to the function F, which satisfies Eq. (12), substitution (9) yields a homogeneous harmonic function of degree n from any solution of equation (12), and there is one-to-one correspondence between the functions F and U. This statement is verified by direct substitution of expression (9) into the 3D Laplace equation, followed by change of variables according to rule (11). A similar technique is used, for example, in [35–39].

Evidently, the transformation of the function U according to rule (8), which should generate a new homogeneous harmonic function V of degree n, is equivalent to transformation of the function F according to the rule

$$G\left(p,q\right)=s\left(p,q\right)F\left(p,q\right)+ +v\left(p,q\right)\frac{\partial F\left(p,q\right)}{\partial p}+w\left(p,q\right)\frac{\partial F\left(p,q\right)}{\partial q} \qquad (13)$$

with some fixed functions

$$s(p,q),\ v(p,q),\ w(p,q),$$

the function G should then satisfy the equation

$$\frac{\partial^2 G\left(p,q\right)}{\partial p^2}+\frac{\partial^2 G\left(p,q\right)}{\partial q^2}+ +\frac{4n\left(n+1\right)}{\left(1+p^2+q^2\right)^2}G\left(p,q\right)=0. \qquad (14)$$

In the end the task is reduced to finding such functions $s(p,q)$, $v(p,q)$, $w(p,q)$ and such indices m and n that would generate the solutions of differential equation (14) after transformation (13) for any solution of differential equation (12). In this case, no additional conditions except Eqs. (12) and (14) are imposed on the functions F and G.

Solution of the problem

After substituting expression (13) into equation (14), we obtain a linear combination of partial derivatives

$$F,\ F_p,\ F_q,\ F_{pp},\ F_{pq},$$
$$F_{qq},\ F_{ppp},\ F_{ppq},\ F_{pqq},\ F_{qqq}$$

(the subscripts denote the partial derivatives taken with respect to the corresponding variables).

Derivatives F_{qq}, F_{pqq}, F_{qqq} are dependent, and they can be expressed in terms of the remaining partial derivatives using Eq. (12). After that, the factors are grouped before the remaining partial derivatives

$$F,\ F_p,\ F_q,\ F_{pp},\ F_{pq},\ F_{ppp},\ F_{ppq},$$

should be zero if the function G is required to satisfy Eq. (14) for any solution of Eq. (12).

The resulting system of equations has the form:

$$v_p - w_q = 0,\ v_q + w_p = 0,$$

$$v_{pp} + v_{qq} + v\frac{4\left(n(n+1)-m(m+1)\right)}{\left(1+p^2+q^2\right)^2} = -2s_p,$$

$$w_{pp} + w_{qq} + w\frac{4\left(n(n+1)-m(m+1)\right)}{\left(1+p^2+q^2\right)^2} = -2s_q,$$

$$s_{pp} + s_{qq} + s\frac{4\left(n(n+1)-m(m+1)\right)}{\left(1+p^2+q^2\right)^2} =$$

$$= -\frac{16m(m+1)}{\left(1+p^2+q^2\right)^3}(pv+qw) +$$

$$+\frac{4m(m+1)}{\left(1+p^2+q^2\right)^2}\left(v_p + w_q\right).$$

Analysis of the obtained overdetermined system of partial differential equations with respect to the unknown functions $s(p,q)$, $v(p,q)$, $w(p,q)$ using the methods [40–46] leads to the following non-degenerate solutions that exhaust all possible cases.

a) $n = -2 - m$ or $n = 1 + m$:

$$\begin{cases} s(p,q) = \dfrac{(m+1)}{1+p^2+q^2}\times \\ \times\left(4c_a p + 4c_b q + c_c\left(1-p^2-q^2\right)\right), \\ v(p,q) = c_c p + 2c_b pq + c_a\left(-1+p^2-q^2\right), \\ w(p,q) = c_c q + 2c_a pq + c_b\left(-1-p^2+q^2\right), \end{cases}$$

where c_a, c_b, c_c are arbitrary constants;

b) $n = -1 + m$ or $n = -m$:

$$\begin{cases} s(p,q) = \dfrac{-m}{1+p^2+q^2}\times \\ \times\left(4c_a p + 4c_b q + c_c\left(1-p^2-q^2\right)\right), \\ v(p,q) = c_c p + 2c_b pq + c_a\left(-1+p^2-q^2\right), \\ w(p,q) = c_c q + 2c_a pq + c_b\left(-1-p^2+q^2\right), \end{cases}$$

where c_a, c_b, c_c are also arbitrary constants;

c) $n = m$ or $n = -1 - m$:

$$\begin{cases} s(p,q) = c, \\ v(p,q) = -c_c q + 2c_b pq + c_a\left(1+p^2-q^2\right), \\ w(p,q) = c_c p + 2c_a pq + c_b\left(1-p^2+q^2\right), \end{cases}$$

where c, c_a, c_b, c_c are also arbitrary constants.

There are also degenerate solutions. The first one has the form

$$s(p,q) = 0,\ v(p,q) = 0,\ w(p,q) = 0$$

and is of no particular interest.

The second solution corresponds to the choice

$$m = 0,\ n = 0,$$

$$\text{or } m = 0,\ n = -1,$$

$$\text{or } m = -1,\ n = 0,$$

$$\text{or } m = -1,\ n = -1.$$

In these cases, Eqs. (12) and (14) turn into two-dimensional Laplace equations, and transformation (13) takes the form

$$G(p,q) = cF(p,q) + v(p,q)F_p(p,q) + \\ + w(p,q)F_q(p,q), \tag{15}$$

where c is constant, and functions $V(p,q)$ and $w(p,q)$ satisfy the Cauchy–Riemann conditions

$$v_p = w_q,\ v_q = -w_p,$$

i.e., are the real and the imaginary part of the analytical function of a complex variable

$$u(p + iq) = v(p,q) + iw(p,q).$$

The physical meaning of Eq. (15) is quite simple: multiplication by a constant transforms the solution of the two-dimensional Laplace equation into a solution of the two-dimensional Laplace equation, and the product of two analytical functions of a complex variable

$$v(p,q) + iw(p,q),\ F_p(p,q) - iF_q(p,q)$$

yields the analytical function of a complex variable whose real and imaginary parts should satisfy the two-dimensional Laplace equation [24–27, 36].

However, this curious degenerate solution is not very interesting as a generator of new homogeneous harmonic functions, since the real and imaginary parts of analytical functions of a complex variable provide enough resources for solving the two-dimensional Laplace equation. All 3D homogeneous harmonic functions with homogeneity degrees of 0 and –1 can be obtained using the Donkin formulas [7–10, 30–34]:

$$V(x,y,z) = H\left(\frac{x}{z+r}, \frac{y}{z+r}\right),$$

$$V(x,y,z) = \frac{1}{r}H\left(\frac{x}{z+r}, \frac{y}{z+r}\right),$$

where is H is the solution of the two-dimensional Laplace equation.

In the end, the following transforming expressions of the form (8) are obtained:

$$V(x,y,z)=U(x,y,z), \tag{16}$$

$$V(x,y,z)=U_x(x,y,z), \tag{17}$$

$$V(x,y,z)=U_y(x,y,z), \tag{18}$$

$$V(x,y,z)=\frac{m}{z}U(x,y,z)-\\ -\frac{x}{z}U_x(x,y,z)-\frac{y}{z}U_y(x,y,z), \tag{19}$$

$$V(x,y,z)=(2m+1)xU(x,y,z)-\\ -r^2U_x(x,y,z), \tag{20}$$

$$V(x,y,z)=(2m+1)yU(x,y,z)-\\ -r^2U_y(x,y,z), \tag{21}$$

$$V(x,y,z)=\frac{-mr^2+(2m+1)z^2}{z}U(x,y,z)+\\ +\frac{xr^2}{z}U_x(x,y,z)+\frac{yr^2}{z}U_y(x,y,z), \tag{22}$$

$$V(x,y,z)=yU_x(x,y,z)-xU_y(x,y,z), \tag{23}$$

$$V(x,y,z)=-\frac{mx}{z}U(x,y,z)+\\ +\frac{x^2+z^2}{z}U_x(x,y,z)+\frac{xy}{z}U_y(x,y,z), \tag{24}$$

$$V(x,y,z)=-\frac{my}{z}U(x,y,z)+\\ +\frac{xy}{z}U_x(x,y,z)+\frac{y^2+z^2}{z}U_y(x,y,z), \tag{25}$$

$$V(x,y,z)=\frac{1}{r^{2m+1}}U(x,y,z), \tag{26}$$

$$V(x,y,z)=\frac{1}{r^{2m-1}}U_x(x,y,z), \tag{27}$$

$$V(x,y,z)=\frac{1}{r^{2m-1}}U_y(x,y,z), \tag{28}$$

$$V(x,y,z)=\frac{m}{zr^{2m-1}}U(x,y,z)-\\ -\frac{x}{zr^{2m-1}}U_x(x,y,z)-\frac{y}{zr^{2m-1}}U_y(x,y,z), \tag{29}$$

$$V(x,y,z)=\frac{(2m+1)x}{r^{2m+3}}U(x,y,z)-\\ -\frac{1}{r^{2m+1}}U_x(x,y,z), \tag{30}$$

$$V(x,y,z)=\frac{(2m+1)y}{r^{2m+3}}U(x,y,z)-\\ -\frac{1}{r^{2m+1}}U_y(x,y,z), \tag{31}$$

$$V(x,y,z)=\frac{(2m+1)z^2-mr^2}{zr^{2m+3}}U(x,y,z)+\\ +\frac{x}{zr^{2m+1}}U_x(x,y,z)+\frac{y}{zr^{2m+1}}U_y(x,y,z), \tag{32}$$

$$V(x,y,z)=\frac{y}{r^{2m+1}}U_x(x,y,z)-\\ -\frac{x}{r^{2m+1}}U_y(x,y,z), \tag{33}$$

$$V(x,y,z)=-\frac{mx}{zr^{2m+1}}U(x,y,z)+\\ +\frac{x^2+z^2}{zr^{2m+1}}U_x(x,y,z)+\frac{xy}{zr^{2m+1}}U_y(x,y,z), \tag{34}$$

$$V(x,y,z)=-\frac{my}{zr^{2m+1}}U(x,y,z)+\\ +\frac{xy}{zr^{2m+1}}U_x(x,y,z)+\frac{y^2+z^2}{zr^{2m+1}}U_y(x,y,z), \tag{35}$$

where $r=\sqrt{x^2+y^2+z^2}$.

The variable z in the denominator typically indicates that the derivative U_z with some nonzero factor is implicitly present in formula; because the Euler equation (3) is used to eliminate the dependent derivative U_z, the derivative subsequently turns into a linear combination of functions U, U_x and U_y). As a result, some expressions can be simplified by eliminating the variable z in the denominator:

$$V(x,y,z)=U_z(x,y,z), \tag{19a}$$

$$V(x,y,z)=(2m+1)zU(x,y,z)-\\ -r^2U_z(x,y,z), \tag{22a}$$

$$V(x,y,z)=zU_x(x,y,z)-xU_z(x,y,z), \tag{24a}$$

$$V(x,y,z)=zU_y(x,y,z)-yU_z(x,y,z), \tag{25a}$$

$$V(x,y,z)=\frac{1}{r^{2m-1}}U_z(x,y,z), \quad (29a)$$

$$V(x,y,z)=\frac{(2m+1)z}{r^{2m+3}}U(x,y,z)-\frac{1}{r^{2m+1}}U_z(x,y,z), \quad (32a)$$

$$V(x,y,z)=\frac{z}{r^{2m+1}}U_x(x,y,z)-\frac{x}{r^{2m+1}}U_z(x,y,z), \quad (34a)$$

$$V(x,y,z)=\frac{z}{r^{2m+1}}U_y(x,y,z)-\frac{y}{r^{2m+1}}U_z(x,y,z). \quad (35a)$$

Conclusion

We have established that the Thomson formula for 3D harmonic functions that are Euler-homogeneous can be generalized if we use linear algebraic form (5) including first-order partial derivatives of the initial function instead of purely algebraic linear expressions. We have provided an exhaustive list of the obtained expressions of the first order, transforming arbitrary homogeneous 3D harmonic functions into new 3D harmonic functions which can be obtained without resorting to change of variables in the arguments of the function U.

Careful checking, however, has revealed that all the expressions obtained are in fact transforming formulas for 3D harmonic functions of a general form from [1], simplified by assuming that the function U and Euler's differential equation (3) for homogeneous functions are homogeneous [21, 22].

Other relations can also be obtained if we consider linear combinations with constant coefficients composed from the basic relations (16)–(35), corresponding to the same degree of homogeneity of the transformed function. Moreover, adding the Euler equation (3) multiplied by an arbitrary homogeneous function of the corresponding degree to any of the Eqs. (16)–(35), we obtain a new transformation.

For example, transformation (7) is such a linear combination; formally speaking, it is fundamentally different from the previously obtained basic Eqs. (16)–(35):

$$\begin{aligned}
L[U]=&\big(a(2m+1)xz+b(2m+1)yz+\\
&+c\big(-mx^2-my^2+(m+1)z^2\big)\big)U_x+\\
&+\big(a(2m+1)xy+b\big(-mx^2+(m+1)y^2-mz^2\big)+\\
&+c(2m+1)yz\big)U_y+\big(a\big((m+1)x^2-my^2-mz^2\big)+\\
&+b(2m+1)xy+c(2m+1)xz\big)U_z=\\
=&(2m+1)(cx+by+az)\big(xU_x+yU_y+zU_z-mU\big)+\\
&+mc\big((2m+1)xU-\big(x^2+y^2+z^2\big)U_x\big)+\\
&+mb\big((2m+1)yU-\big(x^2+y^2+z^2\big)U_y\big)+\\
&+ma\big((2m+1)xU-\big(x^2+y^2+z^2\big)U_z\big).
\end{aligned}$$

However, while such formulas can be considerably different (in the algebraic sense) from the list obtained earlier, they are fully equivalent to the basic Eqs. (16)–(35) serving as generators of new analytical expressions for 3D homogeneous harmonic functions. Similar problems with equivalent mathematical expressions that are not identically equal to each other in the algebraic sense are described, for example, in [47–50].

The calculations given in this paper were carried out using the Wolfram Mathematica software [51].

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