DOI: 10.18721/JPM.12203
УДК 517.51; 517.28; 517.983; 537.213, 537.8

GENERALIZATION OF THE THOMSON FORMULA FOR GENERAL HARMONIC FUNCTIONS

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The paper continues the investigation of electron and ion optical properties of electric and magnetic fields which can be represented in an analytical form. The target of this research is new recipes for generating analytical solutions of 3D Laplace equation, in particular, for generating 3D harmonic functions which are homogeneous in Euler terms. Linear algebraic expressions with first order partial derivatives which generalize the widely known Thomson formula (Kelvin transformation), are analyzed. The paper provides an exhaustive list of symmetric and homogeneous first order differentiating expressions that convert an arbitrary 3D harmonic function into some new 3D harmonic functions. The produced 3D expressions are generalized for the *n*-dimensional case.

Keywords: electrostatic field, magnetostatic field, scalar potential, Laplace's equation, Thomson formula

Citation: Berdnikov A.S., Gall L.N., Gall N.R., Solovyev, K.V. Generalization of the Thomson formula for general harmonic functions, St. Petersburg Polytechnical State University Journal. Physics and Mathematics. 12 (2) (2019) 32–47. DOI: 10.18721/JPM.12203

ОБОБЩЕНИЕ ФОРМУЛЫ ТОМСОНА ДЛЯ ГАРМОНИЧЕСКИХ ФУНКЦИЙ ОБЩЕГО ВИДА

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Статья продолжает цикл работ, посвященный изучению электронно- и ионно-оптических свойств электрических и магнитных полей, представимых в аналитической форме. Целью исследования является поиск альтернативных рецептов для генерирования новых аналитических решений трехмерного уравнения Лапласа и, в частности, для генерирования трехмерных гармонических функций, являющихся однородными по Эйлеру. Рассматриваются обобщения широко известной алгебраической формулы Томсона (преобразование Кельвина), которые используют линейные алгебраические формы с частными производными первого порядка. Приведен исчерпывающий список симметризованных однородных дифференцирующих выражений первого порядка, преобразующих произвольные трехмерные гармонические функции в новые трехмерные гармонические функции. Дано обобщение полученных трехмерных формул на случай произвольного (конечного) числа измерений.

Ключевые слова: электростатическое поле, магнитостатическое поле, скалярный потенциал, уравнение Лапласа, формула Томсона

Ссылка при цитировании: Бердников А.С., Галль Л.Н., Галль Н.Р., Соловьев К.В. Обобщение формулы Томсона для гармонических функций общего вида // Научно-технические ведомости СПбГПУ. Физико-математические науки. 2019. Т. 12. № 2. С. 32–48. DOI: 10.18721/JPM.12203

Introduction

The Thomson formula [1–3] for 3D harmonic functions is a unique tool. Any other formula of this type may differ from the original Thomson formula only by a trivial change of variables, represented as a superposition of shifts, reflections, rotations, and proportional stretching of coordinates. Let us describe in detail how the Thomson formula works.

If $U(x,y,z)$ is an arbitrary harmonic function of three variables, i.e., it satisfies the Laplace equation

$$U_{xx} + U_{yy} + U_{zz} = 0 \quad (1)$$

(from now on we use the subscripts composed of the symbols x, y, z to denote partial derivatives with respect to the corresponding variables), then the function

$$V(x,y,z) = \frac{1}{r} U\left(\frac{x}{r^2}, \frac{y}{r^2}, \frac{z}{r^2}\right), \quad (2)$$

where $r = \sqrt{x^2 + y^2 + z^2}$, is also harmonic [1, 2]. Using Eq. (2) once again, we make the transition from the function $V(x,y,z)$ back to the function $U(x,y,z)$. We can verify that the function V obtained from Eq. (2) is harmonic (provided that the function U is harmonic) using the identity

$$V_{xx} + V_{yy} + V_{zz} \equiv \frac{1}{r^5}\left(U_{xx} + U_{yy} + U_{zz}\right),$$

where the function V is given by expression (2), and the function U is arbitrary.

The change of variables used for the arguments of the function U in Eq. (2) is the inversion in a sphere of unit radius with the center at the origin. Eq. (2) is named the Thomson formula after its author, the eminent British physicist William Thomson, Lord Kelvin [3–5]; sometimes this formula is also called the Kelvin transform [1, 6–10]. Transformation (2) preserves harmonic functions and can be used, in particular, not only for solving boundary problems with the Dirichlet condition (with the interior Dirichlet problem transformed into exterior and vice versa) but also for generating new analytical solutions for scalar potentials of electrostatic and magnetostatic fields, which is convenient in synthesizing electron and ion optical systems [11–13].

Euler-homogeneous electric and magnetic fields are a useful tool for synthesizing a special type of electron and ion optical systems [14–18]. The trajectories of charged particles in Euler-homogeneous electrostatic and magnetostatic fields obey the principle of similarity of trajectories described by Golikov [19, 20]; the unique optical properties of the devices controlling the motion of charged particles and using Euler-homogeneous electric and magnetic fields follow from this principle.

As a rule, these fields are characterized by a scalar electric or magnetic potential which is an Euler-homogeneous (or, more precisely, positively homogeneous) function in the sense given to this term in classical mathematical analysis [21, 22]:

$$\forall\lambda > 0:\ U(\lambda x, \lambda y, \lambda z) \equiv \lambda^k U(x,y,z), \quad (3)$$

where k is the degree of homogeneity of the scalar function (that is not necessarily an integer) and, accordingly, the degree of homogeneity of the field.

Possible exceptions, when the scalar potential of a homogeneous field is not a homogeneous function, were considered in [23].

If $U(x,y,z)$ is a harmonic and Euler-homogeneous function with a degree of homogeneity k, then the harmonic function $V(x,y,z)$ calculated according to rule (2) is also Euler-homogeneous with a degree of homogeneity $(-k - 1)$. Applying transformation (2) again, we make the transition from the function $V(x,y,z)$ back to the function $U(x,y,z)$. Therefore, a harmonic prototype with a degree of homogeneity $(-k - 1)$ necessarily exists for each harmonic function that is Euler-homogeneous with a degree of homogeneity k, so that the function can be obtained from this prototype by Eq. (2). Combined with differentiation with respect to the variables x, y, z, which is a universal method for obtaining new homogeneous harmonic functions with reduced degree of homogeneity [24, 25], the Thomson formula (2) allows to obtain differential/algebraic equations of a general form for 3D homogeneous harmonic functions with any integer-valued degrees of homogeneity [24, 26]. Donkin's formula is used as the starting point for 3D homogeneous harmonic functions of zero degree [16, 17, 24, 27–31]. This issue is considered in more detail in [24, 26].

The common factor $1/r^2$ in Eq. (2) is taken out of the function arguments for Euler-homogeneous functions U satisfying identity (3). Eq. (2) then takes a simplified form (see Thomson's treatise [5], Appendix B to Chapter 1):

$$V(x,y,z) = r^{-2k-1} U(x,y,z). \quad (4)$$

We can verify that if a homogeneous function U is substituted into Eq. (4), function (4) is homogeneous. The function V, calculated in accordance with rule (4), is harmonic, as follows from the identity

$$V_{xx} + V_{yy} + V_{zz} \equiv r^m(U_{xx} + U_{yy} + U_{zz}) + \\ + 2mr^{m-2}(xU_x + yU_y + zU_z - kU) + \\ + m(m + 2k + 1)r^{m-2}U, \tag{5}$$

valid for functions of the form $V(x,y,z) = r^m U(x,y,z)$ with an arbitrary exponent m and an arbitrary function U. Indeed, the right-hand side of identity (5) becomes zero at $m = 0$ and $m = -2k - 1$, since the function U must satisfy the Laplace equation (1) and the Euler differential equation for homogeneous functions [21, 22]:

$$xU_x + yU_y + zU_z = kU. \tag{6}$$

The goal of this study has consisted in finding alternative recipes for generating new analytical solutions of the Laplace equation and, in particular, for generating 3D harmonic functions that Euler-homogeneous.

Uniqueness of Thomson's algebraic formula

Let us consider the transformation of a three-dimensional harmonic function $U(x,y,z)$ in accordance with the rule

$$V(x,y,z) = S(x,y,z)\times \\ \times U\big(f(x,y,z), g(x,y,z), h(x,y,z)\big), \tag{7}$$

where S, f, g, h are some fixed functions.

Here we can confine ourselves to algebraic expressions linear with respect to U, since the Laplace equation (1) has the property of linearity and linear superposition of its solutions with constant coefficients is again a solution.

Let us impose the condition that expression (7) be a harmonic function for any harmonic functions U. After substituting expression (7) into the Laplace equation (1), we obtain a linear combination composed of partial derivatives U, U_x, U_y, U_z, U_{xx}, U_{yy}, U_{zz}, U_{xy}, U_{xz}, U_{yz}. Since the function U is harmonic, the derivative U_{zz} can be expressed in terms of partial derivatives U_{xx} and U_{yy}:

$$U_{zz} = -U_{xx} - U_{yy}. \tag{8}$$

This brings up the question whether the other partial derivatives can be regarded as independent. The answer is yes: the Cauchy problem for the Laplace equation (1) with initial conditions

$$U(x,y,z_0) = U^{(0)}(x,y),$$
$$U_z(x,y,z_0) = U^{(n)}(x,y),$$

set for the plane $z = z_0$, is solvable for any initial values of $U^{(0)}(x,y)$ and $U^{(n)}(x,y)$, at least in some neighborhood of the plane $z = z_0$. For example, this solution could be a Taylor series with respect to the variable z, where all coefficients are uniquely expressed in terms of the functions $U^{(0)}(x,y)$, $U^{(n)}(x,y)$ and their derivatives with respect to x, y. Therefore, the derivatives of the functions $U^{(0)}(x,y)$ and $U^{(n)}(x,y)$ with respect to the variables x and y, calculated at a fixed point, are independent numbers.

Consequently, if there are no additional constraints on the harmonic function U, then the remaining partial derivatives in the final linear combination of partial derivatives of the function U, obtained after substituting condition (8), should be assumed to be independent, and each of the factors grouped before these partial derivatives should be zero. The set of independent partial derivatives includes mixed derivatives of any order with respect to x, y, but only zero and first order with respect to z. This rule holds not only for derivatives whose order is not higher than the second, used in the given linear combination, but also in the general case (see the next section).

As a result, we obtain for the unknown functions S, f, g, h, assuming that $S(x,y,z) \neq 0$, a system of nine partial differential equations:

$$\begin{gathered} f_x g_x + f_y g_y + f_z g_z = 0, \\ f_x h_x + f_y h_y + f_z h_z = 0, \\ g_x h_x + g_y h_y + g_z h_z = 0, \\ f_x^2 + f_y^2 + f_z^2 = h_x^2 + h_y^2 + h_z^2, \\ g_x^2 + g_y^2 + g_z^2 = h_x^2 + h_y^2 + h_z^2, \\ S\left(f_{xx} + f_{yy} + f_{zz}\right) + \\ +2S_x f_x + 2S_y f_y + 2S_z f_z = 0, \\ S\left(g_{xx} + g_{yy} + g_{zz}\right) + \\ +2S_x g_x + 2S_y g_y + 2S_z g_z = 0, \\ S\left(h_{xx} + h_{yy} + h_{zz}\right) + \\ +2S_x h_x + 2S_y h_y + 2S_z h_z = 0, \\ S_{xx} + S_{yy} + S_{zz} = 0. \end{gathered} \tag{9}$$

This system of equations is overdetermined (there are more equations than unknown functions), so, generally speaking, it may not have solutions [32–38]. However, the Thomson formula (2) guarantees that system (9) has non-degenerate non-zero solutions that interest us.

The first five equations of system (9) mean that a one-to-one, continuously differentiable mapping

$$x' = f(x,y,z),\ y' = g(x,y,z),\quad z' = h(x,y,z) \tag{10}$$

is conformal, that is, it locally preserves the angles between the lines at the point where they intersect regardless of the location of these lines, and converts infinitely small segments into proportional infinitely small segments with a proportionality coefficient that does not depend on direction. This mapping preserves the shape of infinitely small figures but does not preserve the length of the lines, their curvature, or the global shape of the figures and, possibly, the orientation of the local basis [39, 40].

The family of conformal transformations for a two-dimensional plane is very diverse and in fact coincides with the family of analytic functions of one complex variable [41–45]. However, this is not the case for three and higher dimensions: the Liouville theorem on conformal mappings in Euclidean spaces (see [46–52]) postulates that the family of conformal mappings coincides with the group of Möbius transformations in these cases [53–55] and no other multidimensional conformal mappings are available. Unfortunately, we have not managed to uncover the elementary proof of this important theorem; evidently, the simplest proof is given in [52].

In general, the Möbius group is a group generated by the following elementary transformations and their superpositions:

a) shifts (parallel translation),

$$x' = x + a,\ y' = y + b,\ z' = z + c;$$

b) three-dimensional rotations around a fixed point [56, section 14.10];

c) reflections with respect to hyperplanes, in particular, elementary symmetries

$$x' = -x,\ y' = -y,\ z' = -z;$$

d) proportional stretching in all coordinates relative to some center

$$x' = kx,\ y' = ky,\ z' = kz;$$

(the origin is used as the center here);

e) inversion relative to the sphere,

$$x' = xr_0^2/(x^2+y^2+z^2),\ y' = yr_0^2/(x^2+y^2+z^2),\quad z' = zr_0^2/(x^2+y^2+z^2)$$

(here r_0 is the radius of the sphere, and the origin is used as the center of the sphere).

Not all of the above transformations are independent. For example, stretching can be replaced by two successive inversions relative to spheres with a common center but with different radii, and reflections relative to hyperplanes $x = 0$, $y = 0$ or $z = 0$ can be replaced by stretching with a scaling factor of −1 combined with 180° rotation relative to one of the coordinate axes.

The Möbius group for the two-dimensional case that is the most typical for practical applications coincides with the group of linear fractional conformal transformations, to which complex conjugate linear fractional conformal transformations are added (anti-conformal, if conformal transformations are understood as those preserving not only local angles but also their direction, i.e., the orientation of the local basis).

We can prove that any element of the Möbius group can be reduced to one of two possible types:

$$\mathbf{r}' = \mathbf{b} + \frac{\lambda A(\mathbf{r}-\mathbf{a})}{|\mathbf{r}-\mathbf{a}|^2}, \tag{11}$$

$$\mathbf{r}' = \mathbf{b} + \lambda A(\mathbf{r}-\mathbf{a}), \tag{12}$$

where **r** is the radius vector (generally speaking, n dimensional) for the initial point; **r'** is the radius vector for the transformed point; **a** is the initial center of the geometric transformation; **b** is the final location of the center of the geometric transformation; λ is the stretch factor (real number); A is an orthogonal matrix satisfying the condition $AA^T = A^TA = E$ and describing rotation in n dimensional space relative to the origin (possibly with a change in the orientation of the local basis if the determinant of the matrix A is equal to −1).

To prove this statement, it is sufficient to verify that the composition of geometric transformation (11) or (12) is again reduced either to reference form (11) or to reference form (12) with each of the elementary transformations of the Möbius group given above.

Three-dimensional geometric transformations of the form (11) or (12) are obviously decomposed into a superposition of elementary transformations in the form of an initial shift, inversion with the center at the origin (for transformation (11)), three-dimensional rotations around the origin [56, section 14.10] (possibly combined with one of the symmetries changing the orientation of the system), proportional stretching relative to the origin and the final shift to the new center. The harmonic function U remains harmonic with a shift, rotation, symmetric reflection and proportional stretching of the arguments, so in these cases the factor $S(x,y,z)$ for Eq. (7) is equal to unity (or, more precisely, to an arbitrary constant, as follows from Eqs. (9)). In case of inversion with the center at the origin, the factor $S(x,y,z)$ for Eq. (7) is determined up to a constant factor in accordance with the Thomson formula (2), and system of equations (9) confirms that this factor is unique.

Successively making these transformations, with the conformal transformation (10) is written either in the form (11) or (12), we obtain the final solution for the problem of finding algebraic formulas of the form (7) which remain harmonic. If, however, an Euler-homogeneous harmonic function is to be transformed into a homogeneous harmonic function, the answer is either the Thomson formula (2) or the identity equality $V(x,y,z) = U(x,y,z)$ up to a rotation and proportional stretching of the arguments x, y, z relative to the origin, and also up to the potential values at all points of space multiplied by a constant. Uniqueness of the Thomson formula is also proved in [8, 10].

Expanded form including first derivatives

The purely algebraic transformation (7) has no other meaningful solutions except the classical Thomson formula (2), leading us to search for other methods of generating new harmonic functions. Let us consider the transformation of the three-dimensional harmonic function $U(x,y,z)$ in accordance with the rule

$$\begin{aligned} V(x,y,z) &= \\ &= S(x,y,z)\cdot U\big(f(x,y,z),g(x,y,z),h(x,y,z)\big)+ \\ &+P(x,y,z)\cdot U_x\big(f(x,y,z),g(x,y,z),h(x,y,z)\big)+ \\ &+Q(x,y,z)\cdot U_y\big(f(x,y,z),g(x,y,z),h(x,y,z)\big)+ \\ &+R(x,y,z)\cdot U_z\big(f(x,y,z),g(x,y,z),h(x,y,z)\big), \end{aligned} \quad (13)$$

where S, P, Q, R, f, g, h are some fixed functions, U is an arbitrary harmonic function.

As before, let us impose that expression (13) be a harmonic function for the given functions S, P, Q, R, f, g, h and any harmonic functions U. To ensure that function (13) is a homogeneous function with homogeneous functions U, we impose that the functions S, P, Q, R, f, g, h also be Euler-homogeneous (it is easy to verify that this requirement is necessary as well as sufficient). To eliminate the freedom of choice in the form of three-dimensional rotations around the origin, which is excessive in our case, similar to Eq. (2), we confine ourselves to the case when $f(x,y,z) = x\varphi(x,y,z)$, $g(x,y,z) = y\varphi(x,y,z)$, $h(x,y,z) = z\varphi(x,y,z)$,
where the common factor $\varphi(x,y,z)$ is an Euler-homogeneous function.

We should however keep in mind that we run the risk of discarding any truly interesting solutions, and not just rotations.

It is convenient to write the homogeneous functions φ, S, P, Q, R in the following form:

$$\begin{aligned} \varphi(x,y,z) &= r^m\omega(x/r,\ y/r), \\ S(x,y,z) &= r^n s(x/r,\ y/r), \\ P(x,y,z) &= r^{n+m+1}u(x/r,\ y/r), \\ Q(x,y,z) &= r^{n+m+1}v(x/r,\ y/r), \\ R(x,y,z) &= r^{n+m+1}w(x/r,\ y/r), \end{aligned} \quad (14)$$

where m is the degree of homogeneity of the common factor for the arguments of the function U; n is the degree of homogeneity of the factor before the function U itself; $\omega(\chi,\eta)$, $s(\chi,\eta)$, $u(\chi,\eta)$, $v(\chi,\eta)$, $w(\chi,\eta)$ are some functions of two variables, unknown so far.

This formulation is a slightly modified universal representation [21, 22] for homogeneous functions of degree k:

$$f\left(x_1,x_2,\ldots,x_n\right) = x_1^k g\left(x_2/x_1,\ldots x_n/x_1\right) \quad (15)$$

and does not lead to loss of admissible solutions. Importantly, the change of variables used for constructing substitution (14) is reversible :

$$\begin{cases} \chi = x\big/\sqrt{x^2+y^2+z^2}, \\ \eta = y\big/\sqrt{x^2+y^2+z^2}, \\ \rho = \sqrt{x^2+y^2+z^2}, \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x = \chi\rho, \\ y = \eta\rho, \\ z = \pm\rho\sqrt{1-\chi^2-\eta^2}. \end{cases}$$

After substituting Eqs. (13) into the Laplace equation (1), we obtain a linear combination with some factors independent of U, composed of partial derivatives

$$U,\ U_x,\ U_y,\ U_z,\ U_{xx},$$
$$U_{yy},\ U_{zz},\ U_{xy},\ U_{xz},\ U_{yz},$$
$$U_{xxx},\ U_{xxy},\ U_{xxz},\ U_{xyy},\ U_{xyz},$$
$$U_{xzz},\ U_{yyy},\ U_{yyz},\ U_{yzz},\ U_{zzz}.$$

Since the function U is harmonic, some of these derivatives can be expressed in terms of others:

$$U_{zz} = -U_{xx} - U_{yy},\ U_{xzz} = -U_{xxx} - U_{xyy},$$
$$U_{zzy} = -U_{xxy} - U_{yyy},\ U_{zzz} = -U_{xxz} - U_{yyz}.$$

The reasoning in the previous section proves that the remaining partial derivatives should be assumed to be independent. Therefore, after substitution into the resulting linear combination of the above expressions for dependent derivatives

$$U_{zz},\ U_{xzz},\ U_{yzz},\ U_{zzz}$$

and finding common factors before the remaining partial derivatives, each of the resulting factors should be identically zero.

The resulting system of partial differential equations with respect to the unknown functions turns out to be overdetermined.

$$\omega(\chi,\eta),\ s(\chi,\eta),\ u(\chi,\eta),\ v(\chi,\eta),\ w(\chi,\eta)$$

It has a rather complicated form, so it is not given here explicitly. By the same logic, we omitted the analysis of compatibility of the obtained system of equations and its solutions, since it is overly cumbersome and, apart from standard technical methods, does not provide any new information. Analysis of this system using the appropriate methods [32–38] yields the following solutions which exhaustively cover all possible cases:

a) when $m = 0$ and $n = -1$ or $m = -2$ and $n = 0$

$$\omega(\chi,\eta) = c,\ s(\chi,\eta) = 0,$$
$$u(\chi,\eta) = c_a,\ v(\chi,\eta) = c_b,\ w(\chi,\eta) = c_c,$$

where c, c_a, c_b, c_c are arbitrary constants;

b) with $m = 0$ and $n = 1$ or $m = n = -2$

$$\omega(\chi,\eta) = c,\ s(\chi,\eta) = (c_a/c)\chi +$$
$$+(c_b/c)\eta + (c_c/c)\sqrt{1-\chi^2-\eta^2},$$
$$u(\chi,\eta) = c_a\left(-1+2\chi^2\right) +$$
$$+2c_b\chi\eta + 2c_c\chi\sqrt{1-\chi^2-\eta^2},$$
$$v(\chi,\eta) = 2c_a\chi\eta + c_b\left(-1+2\eta^2\right) +$$
$$+2c_c\eta\sqrt{1-\chi^2-\eta^2},$$
$$w(\chi,\eta) = 2c_a\chi\sqrt{1-\chi^2-\eta^2} +$$
$$+2c_b\eta\sqrt{1-\chi^2-\eta^2} + c_c\left(1-2\chi^2-2\eta^2\right),$$

where c, c_a, c_b, c_c are arbitrary constants;

c) with $m = n = 0$ or $m = -2$ and $n = -1$

$$\omega(\chi,\eta) = c,\ s(\chi,\eta) = c_e,$$
$$u(\chi,\eta) = c_d\chi + c_c\eta - c_b\sqrt{1-\chi^2-\eta^2},$$
$$v(\chi,\eta) = -c_c\chi + c_d\eta + c_a\sqrt{1-\chi^2-\eta^2},$$
$$w(\chi,\eta) = c_b\chi - c_a\eta + c_d\sqrt{1-\chi^2-\eta^2},$$

where c, c_a, c_b, c_c, c_d, c_e are arbitrary constants.

As a result, we obtain a list of basic formulas transforming the original harmonic functions into new harmonic functions

$$V(x,y,z) = U(x,y,z), \tag{16}$$

$$V(x,y,z) = U_x(x,y,z), \tag{17}$$

$$V(x,y,z) = U_y(x,y,z), \tag{18}$$

$$V(x,y,z) = U_z(x,y,z), \tag{19}$$

$$\begin{gathered} V(x,y,z) = xU(x,y,z) + \\ +\left(x^2-y^2-z^2\right)U_x(x,y,z) + \\ +2xyU_y(x,y,z) + 2xzU_z(x,y,z); \end{gathered} \tag{20}$$

$$\begin{gathered} V(x,y,z) = yU(x,y,z) + \\ 2xyU_x(x,y,z) + \left(-x^2+y^2-z^2\right)\times \\ \times U_y(x,y,z) + 2yzU_z(x,y,z); \end{gathered} \tag{21}$$

$$\begin{gathered} V(x,y,z) = zU(x,y,z) + \\ +2xzU_x(x,y,z) + 2yzU_y(x,y,z) + \\ +\left(-x^2-y^2+z^2\right)U_z(x,y,z); \end{gathered} \tag{22}$$

$$\begin{gathered} V(x,y,z) = xU_x(x,y,z) + \\ +yU_y(x,y,z) + zU_z(x,y,z); \end{gathered} \tag{23}$$

$$V(x,y,z) = yU_x(x,y,z) - xU_y(x,y,z); \tag{24}$$

$$V(x,y,z) = zU_x(x,y,z) - xU_z(x,y,z); \quad (25)$$

$$V(x,y,z) = zU_y(x,y,z) - yU_z(x,y,z); \quad (26)$$

$$V(x,y,z) = \frac{1}{r}U\left(\frac{x}{r^2},\frac{y}{r^2},\frac{z}{r^2}\right); \quad (27)$$

$$V(x,y,z) = \frac{1}{r}U_x\left(\frac{x}{r^2},\frac{y}{r^2},\frac{z}{r^2}\right); \quad (28)$$

$$V(x,y,z) = \frac{1}{r}U_y\left(\frac{x}{r^2},\frac{y}{r^2},\frac{z}{r^2}\right); \quad (29)$$

$$V(x,y,z) = \frac{1}{r}U_z\left(\frac{x}{r^2},\frac{y}{r^2},\frac{z}{r^2}\right); \quad (30)$$

$$\begin{gathered} V(x,y,z) = \frac{x}{r^3}U\left(\frac{x}{r^2},\frac{y}{r^2},\frac{z}{r^2}\right) + \\ + \frac{x^2-y^2-z^2}{r^5}U_x\left(\frac{x}{r^2},\frac{y}{r^2},\frac{z}{r^2}\right) + \\ \frac{2xy}{r^5}U_y\left(\frac{x}{r^2},\frac{y}{r^2},\frac{z}{r^2}\right) + \frac{2xz}{r^5}U_z\left(\frac{x}{r^2},\frac{y}{r^2},\frac{z}{r^2}\right); \end{gathered} \quad (31)$$

$$\begin{gathered} V(x,y,z) = \frac{y}{r^3}U\left(\frac{x}{r^2},\frac{y}{r^2},\frac{z}{r^2}\right) + \frac{2xy}{r^5}U_x(v) + \\ + \frac{-x^2+y^2-z^2}{r^5}U_y\left(\frac{x}{r^2},\frac{y}{r^2},\frac{z}{r^2}\right) + \\ + \frac{2yz}{r^5}U_z\left(\frac{x}{r^2},\frac{y}{r^2},\frac{z}{r^2}\right); \end{gathered} \quad (32)$$

$$\begin{gathered} V(x,y,z) = \frac{z}{r^3}U\left(\frac{x}{r^2},\frac{y}{r^2},\frac{z}{r^2}\right) + \\ + \frac{2xz}{r^5}U_x\left(\frac{x}{r^2},\frac{y}{r^2},\frac{z}{r^2}\right) + \\ + \frac{2yz}{r^5}U_y\left(\frac{x}{r^2},\frac{y}{r^2},\frac{z}{r^2}\right) + \\ + \frac{-x^2-y^2+z^2}{r^5}U_z\left(\frac{x}{r^2},\frac{y}{r^2},\frac{z}{r^2}\right); \end{gathered} \quad (33)$$

$$\begin{gathered} V(x,y,z) = \frac{x}{r^3}U_x\left(\frac{x}{r^2},\frac{y}{r^2},\frac{z}{r^2}\right) + \\ + \frac{y}{r^3}U_y\left(\frac{x}{r^2},\frac{y}{r^2},\frac{z}{r^2}\right) + \frac{z}{r^3}U_z\left(\frac{x}{r^2},\frac{y}{r^2},\frac{z}{r^2}\right); \end{gathered} \quad (34)$$

$$\begin{gathered} V(x,y,z) = \frac{y}{r^3}U_x\left(\frac{x}{r^2},\frac{y}{r^2},\frac{z}{r^2}\right) - \\ - \frac{x}{r^3}U_y\left(\frac{x}{r^2},\frac{y}{r^2},\frac{z}{r^2}\right); \end{gathered} \quad (35)$$

$$\begin{gathered} V(x,y,z) = \frac{z}{r^3}U_x\left(\frac{x}{r^2},\frac{y}{r^2},\frac{z}{r^2}\right) - \\ - \frac{x}{r^3}U_z\left(\frac{x}{r^2},\frac{y}{r^2},\frac{z}{r^2}\right); \end{gathered} \quad (36)$$

$$\begin{gathered} V(x,y,z) = \frac{z}{r^3}U_y\left(\frac{x}{r^2},\frac{y}{r^2},\frac{z}{r^2}\right) - \\ - \frac{y}{r^3}U_z\left(\frac{x}{r^2},\frac{y}{r^2},\frac{z}{r^2}\right), \end{gathered} \quad (37)$$

where $r = \sqrt{x^2 + y^2 + z^2}$.

Eqs. (16)–(19) are trivial but we still included them in this list for formal reasons. The origin of Eqs. (20)–(22) is not obvious at first glance but they are evidently obtained from the Thomson formula (2) after it is differentiated with respect to one of the variables x, y, z and the Thomson transformation (2) restoring the initial form of the arguments of the function U is applied again.

Eqs. (24)–(26) are mentioned in monograph [24] but for some reason only in relation to homogeneous harmonic functions of zero degree. We have not actually encountered Eq. (23), as well as Eqs. (20)–(22) before (which is not to say of course that there are no such references).

Eqs. (27)–(37) are obtained from Eqs. (16)–(26) using the Thomson transform. In particular, Eq. (27) is obtained from the identical transformation (16) and, by virtue of this, coincides with Eq. (2).

Unlike the original Thomson formula, repeated application of differentiating transformations (16)–(37) does not restore the initial form of the transformed functions. However, some combinations of transformations (16)–(37) may turn out to be identical or again one the formulas from the given set. The reason for this is that the higher derivatives of the function U, appearing from combining differentiating transformations (16)–(37), can eventually be reduced, since the function U satisfies the Laplace equation.

Eqs. (16) and (23)–(26) for Euler-homogeneous functions preserve the degree of homogeneity of the function. Eqs. (17)–(19) reduce the degree of homogeneity of the function by one, and Eqs. (20)–(22) increase the degree of homogeneity of the function by one. Accordingly, homogeneous functions of degree k are transformed by Eqs. (27) and (34)–(37) into homogeneous functions of degree $(-k-1)$, into homogeneous functions of degree $(-k)$ by Eqs. (28)–(30), and into homogeneous functions of degree $(-k-2)$ by Eqs. (31)–(33).

The same as Eq. (16), Eqs. (23) and (34) are completely useless for Euler-homogeneous harmonic functions. It follows from the differential Euler relation (6) for homogeneous functions [21, 22] that applying Eq. (23) yields the same homogeneous harmonic function, only multiplied by a constant (degree of homogeneity). Accordingly, applying Eq. (34) is equivalent to applying Eq. (27) multiplied by a constant.

Generalization to the case of *n* variables

It is known that the Thomson formula (2) for 3D harmonic functions can be generalized to the multidimensional case [7, 9]. If

$$V\left(x_1,x_2,\dots x_n\right)=\frac{1}{r^{n-2}}U\left(\frac{x_1}{r^2},\frac{x_2}{r^2},\cdots,\frac{x_n}{r^2}\right), \quad (38)$$

where $r=\sqrt{x_1^2+x_2^2+\cdots+x_n^2}$, and the function U is harmonic (i.e., it satisfies the n dimensional Laplace equation), then the function V is also harmonic. The result of substituting the function

$$V\left(x_1,x_2,\dots x_n\right)=\frac{1}{r^m}U\left(\frac{x_1}{r^2},\frac{x_2}{r^2},\cdots,\frac{x_n}{r^2}\right)=$$
$$=\frac{1}{r^m}U^*\left(x_1,x_2,\cdots,x_n\right)$$

into the Laplace equation is a chain of equalities

$$\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}\left(\frac{1}{r^m}U^*\right)=U^*\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}\left(\frac{1}{r^m}\right)+$$
$$+2\sum_{i=1}^{n}\frac{\partial U^*}{\partial x_i}\frac{\partial}{\partial x_i}\left(\frac{1}{r^m}\right)+\frac{1}{r^m}\sum_{i=1}^{n}\frac{\partial^2 U^*}{\partial x_i^2};$$
$$\frac{\partial U^*}{\partial x_i}=\sum_{k=1}^{n}\frac{\partial U}{\partial x_k}\left(\frac{-2x_i x_k}{r^4}\right)+\frac{\partial U}{\partial x_i}\frac{1}{r^2};$$
$$\frac{\partial^2 U^*}{\partial x_i^2}=\sum_{k=1}^{n}\frac{\partial U}{\partial x_k}\left(\frac{-2x_k}{r^4}+\frac{8x_i^2 x_k}{r^6}\right)+\frac{\partial U}{\partial x_i}\left(\frac{-2x_i}{r^4}\right)+$$
$$+\sum_{s=1}^{n}\sum_{k=1}^{n}\frac{\partial^2 U}{\partial x_k\partial x_s}\left(\frac{4x_i^2 x_k x_s}{r^8}\right)+\sum_{k=1}^{n}\frac{\partial^2 U}{\partial x_k\partial x_i}\left(\frac{-2x_i x_k}{r^6}\right)+$$
$$+\frac{\partial U}{\partial x_i}\left(\frac{-2x_i}{r^4}\right)+\sum_{k=1}^{n}\frac{\partial^2 U}{\partial x_k\partial x_i}\left(\frac{-2x_i x_k}{r^6}\right)+\frac{\partial^2 U}{\partial x_i^2}\frac{1}{r^4};$$
$$\frac{\partial}{\partial x_i}\left(\frac{1}{r^m}\right)=\frac{-mx_i}{r^{m+2}};$$
$$\frac{\partial^2}{\partial x_i^2}\left(\frac{1}{r^m}\right)=\frac{-m}{r^{m+2}}+\frac{m\left(m+2\right)x_i^2}{r^{m+4}},$$

using, as arguments for function U, the values

x_1/r^2, x_2/r^2, ..., x_n/r^2.

Ultimately, because $\sum_{i=1,n}x_i^2=r^2$, we obtain the identity

$$\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}\left(\frac{U^*}{r^m}\right)=\frac{m\left(m+2-n\right)}{r^{m+2}}U+$$
$$+\frac{2\left(m+2-n\right)}{r^{m+4}}\sum_{k=1}^{n}x_k\frac{\partial U}{\partial x_k}+\frac{1}{r^{m+4}}\sum_{i=1}^{n}\frac{\partial^2 U}{\partial x_i^2},$$

whose right-hand side for harmonic functions U becomes zero for $m=n-2$. Notably, Eq. (38) remains valid, including with $n=2$, when it turns out to be a particular case of a conformal transformation of a plane (or, more precisely, anti-conformal transform, i.e., with a change in the direction in which angles are measured):

$$V\left(x,y\right)=U\left(\frac{x}{x^2+y^2},\frac{y}{x^2+y^2}\right).$$

Formulas including partial derivatives of the first order from the previous section are also transferred to the multidimensional case:

$$V\left(x_1,x_2,\dots x_n\right)=U\left(x_1,x_2,\dots x_n\right); \quad (39)$$

$$V\left(x_1,x_2,\dots x_n\right)=\frac{\partial U\left(x_1,x_2,\dots x_n\right)}{\partial x_i}; \quad (40)$$

$$V\left(x_1,x_2,\dots x_n\right)=\left(n-2\right)x_i U\left(x_1,x_2,\dots x_n\right)+$$
$$+\left(2x_i^2-r^2\right)\frac{\partial U\left(x_1,x_2,\dots x_n\right)}{\partial x_i}+ \quad (41)$$
$$+\sum_{k\neq i}2x_i x_k\frac{\partial U\left(x_1,x_2,\dots x_n\right)}{\partial x_k};$$

$$V\left(x_1,x_2,\ldots x_n\right)=\sum_k x_k\frac{\partial U\left(x_1,x_2,\ldots x_n\right)}{\partial x_k}; \quad (42)$$

$$V\left(x_1,x_2,\ldots x_n\right)=x_i\frac{\partial U\left(x_1,x_2,\ldots x_n\right)}{\partial x_j}- -x_j\frac{\partial U\left(x_1,x_2,\ldots x_n\right)}{\partial x_i}; \quad (43)$$

$$V\left(x_1,x_2,\ldots x_n\right)=\frac{1}{r^{n-2}}U\left(\frac{x_1}{r^2},\frac{x_2}{r^2},\cdots,\frac{x_n}{r^2}\right); \quad (44)$$

$$V\left(x_1,x_2,\ldots x_n\right)=\frac{1}{r^{n-2}}\frac{\partial U}{\partial x_i}\left(\frac{x_1}{r^2},\frac{x_2}{r^2},\cdots,\frac{x_n}{r^2}\right); \quad (45)$$

$$V\left(x_1,x_2,\ldots x_n\right)=\frac{(n-2)x_i}{r^n}U\left(\frac{x_1}{r^2},\frac{x_2}{r^2},\cdots,\frac{x_n}{r^2}\right)+ +\frac{\left(2x_i^2-r^2\right)}{r^{n+2}}\frac{\partial U}{\partial x_i}\left(\frac{x_1}{r^2},\frac{x_2}{r^2},\cdots,\frac{x_n}{r^2}\right)+ +\sum_{k\neq i}\frac{2x_ix_k}{r^{n+2}}\frac{\partial U}{\partial x_k}\left(\frac{x_1}{r^2},\frac{x_2}{r^2},\cdots,\frac{x_n}{r^2}\right); \quad (46)$$

$$V\left(x_1,x_2,\ldots x_n\right)= =\sum_k\frac{x_k}{r^n}\frac{\partial U}{\partial x_k}\left(\frac{x_1}{r^2},\frac{x_2}{r^2},\cdots,\frac{x_n}{r^2}\right); \quad (47)$$

$$V\left(x_1,x_2,\ldots x_n\right)=\frac{x_i}{r^n}\frac{\partial U}{\partial x_j}\left(\frac{x_1}{r^2},\frac{x_2}{r^2},\cdots,\frac{x_n}{r^2}\right)- -\frac{x_j}{r^n}\frac{\partial U}{\partial x_i}\left(\frac{x_1}{r^2},\frac{x_2}{r^2},\cdots,\frac{x_n}{r^2}\right), \quad (48)$$

where $r=\sqrt{x_1^2+x_2^2+\cdots+x_n^2}$.

In contrast to the three-dimensional case, we cannot be certain that the given formulas exhaust the entire list of symmetric homogeneous algebraic formulas including first derivatives transforming an arbitrary n dimensional harmonic function into a new n dimensional harmonic function.

Eq. (46) is obtained by differentiating Eq. (38) with respect to the variable x_i. Eq. (41) is obtained from Eq. (46) using substitution (38). Substitution (38) is also used to obtain Eq. (47) from Eq. (42) and Eq. (48) from Eq. (43). Finally, we can verify that Eq. (42) is valid by using the identity

$$\sum_i\frac{\partial^2 V\left(x_1,\ldots x_n\right)}{\partial x_i^2}\equiv\sum_i\frac{\partial^2 U\left(x_1,\ldots x_n\right)}{\partial x_i^2}+ +\sum_k\left(x_k\frac{\partial}{\partial x_k}\left(\sum_i\frac{\partial^2 U\left(x_1,\ldots x_n\right)}{\partial x_i^2}\right)\right),$$

which is satisfied for any functions V of the form (42), that Eq. (43) is valid using the identity

$$\sum_k\frac{\partial^2 V\left(x_1,\ldots x_n\right)}{\partial x_k^2}\equiv \equiv x_i\frac{\partial}{\partial x_j}\left(\sum_k\frac{\partial^2 U\left(x_1,\ldots x_n\right)}{\partial x_k^2}\right)- -x_j\frac{\partial}{\partial x_i}\left(\sum_k\frac{\partial^2 U\left(x_1,\ldots x_n\right)}{\partial x_k^2}\right),$$

which is satisfied for any functions of V form (43).

Conclusion

The study presents an exhaustive list of homogeneous symmetric differentiating expressions of the first order transforming arbitrary 3D harmonic functions into new 3D harmonic functions. We have generalized the formulas obtained to the case of an arbitrary number of measurements.

There are similar formulas using partial derivatives of higher orders. In particular, such formulas can be obtained by multiple differentiation of the Thomson formula (2) with respect to the variables x, y and z, and by superposition of first-order differential transformations obtained in this study (16)–(37). Compiling a complete list of transforming formulas with derivatives of higher orders is beyond the scope of this study; it is a task that presents considerable technical difficulties and, in our opinion, has little practical meaning.

It should be borne in mind for transformations with derivatives of higher orders that some partial derivatives of the second order and higher are expressed through each other for harmonic functions. Any transforming formula can be supplemented with the three-dimensional Laplace equation (1) with an arbitrary factor, in original form or after it is differentiated with respect to x, y or z the necessary number of times. This does not change the nature of the transformation (only its algebraic form) and, while its basic property to transform the original harmonic functions into new harmonic ones is preserved, additional analytical expressions for 3D harmonic functions are not generated.

Notably, the calculations implicitly led to the assumption that substitution (10) is nondegenerate (reversible). It is possible that there are some additional solutions using changes of degenerate variables, not considered in this study. However, such degenerate transformations of harmonic functions apparently have little practical value, even if they do exist. For example, substituting constants for the arguments of any harmonic function, we obtain a constant, which is, of course, a harmonic function formally speaking, but it is completely useless for practical purposes.

The weak point of the analysis carried out is the assumption that the change of variables has a symmetric form,

$$f(x, y, z) = x\varphi(x, y, z),$$
$$g(x, y, z) = y\varphi(x, y, z),$$
$$h(x, y, z) = z\varphi(x, y, z),$$

while all functions involved in the differential/algebraic Eq. (13) are Euler-homogeneous. We deliberately limited the list of solutions (16)–(37) to homogeneous symmetric linear differential expressions of the first order. There are perhaps additional differential/algebraic expressions of the form (13), different from solutions (16)–(37) and free from this constraint, which also transform 3D harmonic functions into new 3D harmonic functions. However, comprehensive analysis of such an extended [problem [57–60] is well beyond the goals set in our study.

We should also note that the approach described has proved useful and efficient for other equations of mathematical physics beside the Laplace equation (1). For example, similar differential transformations were considered in [61] for the multidimensional heat equation. Unfortunately, the results obtained in [61] cannot be directly transferred to the Laplace equation, even though it is the steady-state limit of the heat equation. The reason for this is that the transformations used in [61] explicitly include time in such a manner that stationary solutions are transformed into non-stationary.

Acknowledgments

This study continues the series of works considering the electron and ion optical properties of Euler-homogeneous electric and magnetic fields that can be represented in analytical form. This series carries on the tradition of analytical studies into the properties of electron and ion optical systems initiated by our mentor Yuri Konstantinovich Golikov (1942–2013), an outstanding theoretical physicist who worked at the Department of Physical Electronics of Peter the Great St. Petersburg Polytechnic University and generously shared with his students his deep encyclopedic knowledge of fundamental mathematical results obtained by prominent scholars of the 18th, 19th and early 20th centuries, more or less forgotten in this computer age.

We are deeply indebted to the creators, employees and sponsors of the websites of Bibliothиque Nationale de France [62] and Proceedings of the Royal Society of London [63], whose dedicated efforts made it possible for us to freely access unique archival materials [3 28, 29], for which we express our heartfelt gratitude. We also extend our gratitude to the creators, employees, and sponsors of the Numdam Digital Library [64] for providing open access to rare mathematical publications from French mathematical journals, in particular, the article published in 1883 [65]. Furthermore, the data we obtained from HathiTrust Digital Library [66] and Das Göttinger Digitalisierungszentrum (GDZ) [67] were incredibly useful for our studies of archival sources, providing access to the digitized version of Cambridge and Dublin Mathematical Journal, published at the end of the 19th century and containing the original publications by William Thomson and George Boole on the given subject. We express our sincere gratitude to these public organizations and foundations, selflessly engaged in preserving Europe's unique historical scientific heritage and granting open access to it.

The calculations given in this paper were carried out using the Wolfram Mathematica software [68].

This study was partially carried out within the framework of State Task No. 075-00780-19-00 of the Institute for Analytical Instrumentation of the Russian Academy of Sciences.

REFERENCES

1. **Sretensky L.N.,** Teoriya niyutonovskogo potentsiala [Theory of the Newtonian potential], Moscow, Leningrad, OGIZ-GITTL, 1946.

2. **Golikov Yu.K.,** Analytical ways of describing harmonic functions, Vestnik of Aktobe's K. Zhubanov Regional State University, Physics and Mathematics. (2(44)) (2016) 165–181.

3. **Thomson W.,** Extraits de deux Lettres adressйes a M. Liouville, Journal de mathйmatiques pures et appliquйs. 12 (1847) 256–264.

4. **Thomson W. (Lord Kelvin), Tait P.G.,** Treatise on natural philosophy, 2nd Ed., Part 1, University Press, Cambridge, 1912.

5. **Thomson W. (Lord Kelvin), Tait P.G.,** Treatise on natural philosophy, 2nd Ed., Part 2, University Press, Cambridge, 1912.

6. **Smirnov V.I.,** A course of higher mathematics, Vol. 4: Integral equations and partial differential equations, Pergamon Press, Oxford, 1964.

7. **Vladimirov V.S.,** Uravneniya matematicheskoy fiziki [Equations of mathematical physics], Mir Publishers, Moscow, 1984.

8. **Kellogg O.D.**, Foundations of potential theory, Springer-Verlag, Berlin, Heidelberg, New-York, 1967.

9. **Wermer J.,** Potential theory, 2nd Ed., Ser. "Lecture Notes in Mathematics", Vol. 408, Springer-Verlag, Berlin, Heidelberg, New-York, 1981.

10. **Helms L.L.** Potential theory, 2nd Ed., Springer, London, Heidelberg, New-York, Dordrecht, 2014.

11. **Golikov Yu.K., Utkin K.G., Cheparuhin V.V.,** Raschet elementov elektrostaticheskikh elektronno-opticheslikh sistem [Design of elements of electron optical systems], Leningrad Polytechnic Inst. Publishing, Leningrad, 1984.

12. **Golikov Yu.K., Solovyov K.V.,** Elektrostaticheskiye ionnyye lovushki [Electrostatic ion traps], Saint-Petersburg Polytechnic University Publishing, Saint-Petersburg, 2008.

13. **Golikov Yu.K., Krasnova N.K.,** Teoriya synteza elektrostaticheskikh energoanalizatorov [Theory of designing of electrostatic energy analyzers], Saint-Petersburg Polytechnic University Publishing, Saint-Petersburg, 2010.

14. **Golikov Yu.K., Krasnova N.K.,** Application of electric fields uniform in the Euler sense in electron spectrography, Technical Physics. 56 (2) (2011) 164–170.

15. **Krasnova N.K.,** Theory and synthesis of disperging and focusing electron optical media, Dr. Sci. Thesis. St. Petersburg, 2013. (in Russian).

16. **Gabdullin P.G., Golikov Yu.K., Krasnova N.K., Davydov S.N.,** The use of Donkin's formula in the theory of energy analyzers, I, Technical Physics. 45 (2) (2000) 232–235.

17. **Gabdullin P.G., Golikov Yu.K., Krasnova N.K., Davydov S.N.,** The use of Donkin's formula in the theory of energy analyzers, II, Technical Physics. 45 (3) (2000) 330–333.

18. **Golikov Yu.K., Krasnova N.K.,** Analytical structures of generalized homogeneous electrical spectrographic media, Nauchnoe priborostroenie (Scientific Instrumentation). 24 (1) (2014) 50–58.

19. **Golikov Yu.K., Krasnova N.K.,** Generalized similarity principle of similarity in electron spectrography, Prikladnaya fizika (Applied Physics). (2) (2007) 5–11.

20. **Averin I.A., Berdnikov A.S., Gall N.R.,** The principle of similarity of trajectories for the motion of charged particles with different masses in electric and magnetic fields that are homogeneous in Euler terms, Technical Physics Letters. 43 (2) (2017) 156–158.

21. **Fichtenholz G.M.,** Differential- und Integralrechnung, I, VEB Deutscher Verlag der Wissenschaften, Berlin. 1968.

22. **Smirnov V.I.**, A course of higher mathematics, Vol. I: Elementary Calculus, Pergamon Press, Oxford, 1964.

23. **Berdnikov A.S., Averin I.A., Krasnova N.K., Solovyev K.V.,** On homogeneity of scalar and vector potentials of electric and magnetic fields, homogeneous in Euler terms, Uspekhi Prikladnoi Fiziki (Advances in Applied Physics). 5 (1) (2017) 10–27.

24. **Hobson E.W.,** The theory of spherical and ellipsoidal harmonics, University Press, Cambridge, 2012.

25. **Berdnikov A.S., Krasnova N.K., Solovyev K.V.,** The theorem on the differentiation of three-dimensional electric and magnetic potentials, homogeneous in Euler terms, Nauchnoe priborostroenie (Scientific Instrumentation). 27 (3) (2017) 107–119.

26. **Berdnikov A.S., Averin I.A., Krasnova N.K., Solovyev K.V.,** General formulas for three-dimensional electric and magnetic potentials homogeneous in Euler terms with an integer order of homogeneity, Nauchnoe priborostroenie (Scientific Instrumentation). 2016. Vol. 26 (4) (2016) 13–30.

27. **Whittaker E.T., Watson G.N.,** A course of modern analysis, University Press, Cambridge, 1950.

28. **Donkin W.F.,** On the equation of Laplace's functions &c., Philosophical Transactions of the

Royal Society of London. 147 (1857) 43–57.
29. **Donkin W.F.,** On the equation of Laplace‘s functions &c., Philosophical Proceedings of the Royal Society of London. 8 (1856–1857) 307–310.
30. **Golikov Yu.K.,** Resheniye zadachi Koshy dlya odnorodnykh garmonicheskikh potentsialov nulevoy plotnosty [Solution of the Cauchy problem for homogeneous harmonic potentials of a zero power], Vestnik of Aktobe’s K. Zhubanov Regional State University, Physics and Mathematics. (2 (44)) (2016) 59–62.
31. **Golikov Yu.K., Berdnikov A.S., Antonov A.S., et al.,** Application of the Donkin formula in the theory of electrostatic prisms, Technical Physics. 63 (11) (2018) 1659–1666.
32. **Gьnther N.M.,** Integrirovaniye uravneniy v chastnykh proizvodnykh pervogo poryadka [Integration of first-order partial differential equations], ONTI, Leningrad, Moscow, 1934.
33. **Tricomi F.G.,** Lezioni sulle equazioni a derivative parziali, Editrice Gherono, Torino, 1954.
34. **Rashevsky P.K.,** Geometricheskaya teoriya uravneniy s chastnymy proizvodnymy [Geometric theory of partial differential equations], OGIZ, Moscow, Leningrad, 1947.
35. **Finikov S.P.,** Metod vneshnikh form Kartana v differentsialnoy geometriy [Cartan’s method of exterior forms in differential geometry], OGIZ, Moscow, Leningrad, 1948.
36. **Cartan Й.,** Les systиmes diffйrentiels extйrieurs et leurs applications gйomйtriques, Ser. “Actualitйs Scientifiques et Industrielles”, No. 994, Hermann, Paris, 1945.
37. **Shcherbakov R.N.,** Osnovy metoda vneshnikh form i lineuchatoy differentsialnoy geometriy [Fundamentals of the method of external forms and differential line geometry], Tomsk State University Publishing House, Tomsk, 1973.
38. **Kaptsov O.V.,** Metody integrirovaniya uravneniy s chastnymy proizvodnymy [Methods for integrating partial differential equations], Fizmatlit, Moscow, 2009.
39. **Janushauskas A.,** Harmonic mappings as defined by M.A. Laurentiev, In: Partial Differential Equations, Banach Center Publications. 10 (1) (1983) 213–241.
40. **Janushauskas A.I.,** Tryokhmernyye analogui konformnykh otobrazheniy [Three-dimensional analogues of conformal mappings], Nauka, Novosibirsk, 1982.
41. **Koppenfels W., Stallmann F.,** Praxis der Konformen Abbildung, Springer-Verlag, Berlin, Göttingen, Heidelberg, 1959.
42. **Lavrent’ev M.A., Shabat B.V.,** Methoden der komplexen Funktionen-theorie, Verlag Wissenshaft, 1969.
43. **Markushevich A.I.,** Theory of functions of a complex variable, Vol. 1–3, Providence, US, Amer. Math. Society Publ., 2005.
44. **Evgrafov M.A.,** Analytic functions, Dover Publ., 1978.
45. **Hurwitz A., Courant R.,** Vorlesungen Uber Allgemeine Funktionentheorie und Elliptische Funktionen, Springer, Berlin. 1964.
46. **Klein F.,** Vorlesungen ьber höhere Geometrie, Verlag von Julius Springer, Berlin, 1926.
47. **Reshetnyak Yu.G.,** Teorema Liuvilya o konformnykh otobrazheniyakh pri minimalnykh predpolozheniyakh regulyarnosty [Liouville’s theorem on conformal mappings for minimal regularity assumptions], Siberian Mathematical Journal. 8 (4) (1967) 631–634.
48. **Flanders H.,** Liouville’s theorem on conformal mapping, Journal of Mathematics and Mechanics. 15 (1) (1966) 157–161.
49. **Phillips R.,** Liouville’s theorem, Pacific Journal. 28 (2) (1969) 397–405.
50. **Blair D.E.,** Inversion theory and conformal mapping, Ser. “Student Mathematical Library”, Vol. 9, American Mathematical Society, 2000.
51.“Liouville’s theorem (conformal_ mappings)”, URL: https:// en.wikipedia. org/ wiki/.
52. **Rashevsky P.K.,** Introduction to Riemannian geometry and tensor analysis, ONTI NKTP USSR, Moscow, Leningrad, 1936.
53. **Kobayashi S.,** Transformation groups in differential geometry, Ser. “Classics in Mathematics”, Vol. 70, New York: Springer-Verlag, Berlin, 1995.
54. **Wilker J.B.,** Inversive geometry, In: Davis C., Grьnbaum B., Sherk F.A. (Eds.) The Geometric Vein, Springer, New York (1981) 379–442.
55. “Möbius transformation”, Section “Higher dimensions”, URL: https://en.wikipedia.org/ wiki/.
56. **Korn G.A., Korn T.M.,** Mathematical handbook for scientists and engineers, Definitions, theorems and formulas for reference and review, 2nd Ed., McGraw-Hill Book Company, New York, San Francisco, Toronto, London, Sydney, 1968.
57. **Bulyanitsa A.L., Kurochkin V.E.,** Studying ordering processes in open systems (on the example of pattern evolution in colonies of imperfect mycelial fungi), Nauchnoye priborostroyeniye (Scientific Instrumentation). 10 (2) (2000) 43–49.
58. **Bulyanitsa A.L., Evstrapov A.A.,**

Rudnitskaya G.E., Calculation of microscale system channel parameters by the method of moments, Nauchnoye priborostroyeniye (Scientific Instrumentation). 13 (4) (2003) 28–40.

59. **Evstrapov A.A., Bulyanitsa A.L., Rudnitskaya G.E., et al.,** Characteristic features of digital signal filtering algorithms as applied to electrophoresis on a microchip, Nauchnoye priborostroyeniye (Scientific Instrumentation). 13 (2) (2003) 57–63.

60. **Bulyanitsa A.L.,** Mathematical modeling in microfluidics: basic concepts, Nauchnoye priborostroyeniye (Scientific Instrumentation). 15 (2) (2005) 51–66.

61. **Brzezina M., Šimůnková M.,** On the harmonic morphism for the Kolmogorov type operators, In: Král J., Lukeš J., Netuka I., Veselý J. (Eds.), Potential Theory – ICPT-94, Walter de Gruyter, Berlin, New York (1996) 341–358.

62. Bibliothèque nationale de France, URL: http://gallica.bnf.fr.

63. Proceedings of the Royal Society of London, URL: http://rspl.royalsocietypublishing.org.

64. Numdam, the French digital mathematics library, URL: http://www.numdam.org/.

65. **Floquet G.**, Sur les equations différentielles linéaires a coefficients périodiques, Annales scientifiques de l'École Normale Supérieure, 2e seriй. 12 (1883) 47–88.

66. HathiTrust Digital Library, URL: https://www.hathitrust.org/.

67. Das Göttinger Digitalisierungszentrum (GDZ), URL: https://gdz.sub.uni-goettingen.de/.

68. Wolfram Mathematica, URL: http://wolfram.com/mathematica/.

Received 21.03.2019, accepted 15.04.2019.

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1. **Сретенский Л.Н.** Теория ньютоновского потенциала. Москва, Ленинград: ОГИЗ-ГИТТЛ, 318 .1946 с.

2. **Голиков Ю.К.** Аналитические способы описания гармонических функций // Вестник Актюбинского регионального государственного университета им. К. Жубанова. Физико-математические науки. 44) 2 № .2016). С. 165–181.

3. **Thomson W.** Extraits de deux Lettres adressées a M. Liouville // Journal de Mathématiques Pures et Appliquées. 1847. Tome XII. Pp. 256–264.

4. **Томсон У. (лорд Кельвин), Тэт П.Г.** Трактат по натуральной философии. Ч. I. Москва, Ижевск: НИЦ «Регулярная и хаотическая динамика», 2010. 572 с.

5. **Томсон У. (лорд Кельвин), Тэт П.Г.** Трактат по натуральной философии. Ч. II. Москва, Ижевск: НИЦ «Регулярная и

хаотическая динамика», 2011. 560 с.
6. **Смирнов В.И.** Курс высшей математики. Т. 4. Ч. 2. М.: Наука, 297 .1981 с.
7. Владимиров В.С. Уравнения математической физики. М.: Наука, 1981. 512 с.
8. **Kellogg O.D.** Foundations of potential theory. Berlin, Heidelberg, New-York: Springer-Verlag, 1967. 386 p.
9. **Уэрмлер Дж.** Теория потенциала. М.: Мир, 1980. 134 с.
10. **Helms L.L.** Potential theory. 2nd Ed. London, Heidelberg, New-York, Dordrecht: Springer, 2014. 485 p.
11. **Голиков Ю.К., Уткин К.Г., Чепарухин В.В.** Расчет элементов электростатических электронно-оптических систем. Ленинград: Изд-во ЛПИ, 79 .1984 с.
12. **Голиков Ю.К., Соловьёв К.В.** Электростатические ионные ловушки. СПб.: Изд-во Политехнического ун-та, 152 .2008 с.
13. **Голиков Ю.К., Краснова Н.К.** Теория синтеза электростатических энергоанализаторов. СПб.: Изд-во Политехнического ун-та, 409 .2010 с.
14. **Голиков Ю.К., Краснова Н.К.** Электрические поля, однородные по Эйлеру, для электронной спектрографии // Журнал технической физики. 2011. Т. 2 № .81. С. 9—15.
15. **Краснова Н.К.** Теория и синтез диспергирующих и фокусирующих электронно-оптических сред. Дис.... докт. физ.-мат. наук. 01.04.04. СПб.: СПбГПУ, 2013. 259 с.
16. **Габдуллин П.Г., Голиков Ю.К., Краснова Н.К., Давыдов С.Н.** Применение формулы Донкина в теории энергоанализаторов. I. // Журнал технической физики. 2000. Т. 70. № 2. С. 91—94.
17. **Габдуллин П.Г., Голиков Ю.К., Краснова Н.К., Давыдов С.Н.** Применение формулы Донкина в теории энергоанализаторов. II. // Журнал технической физики. 2000. Т. 70. № 3. С. 44—47.
18. **Голиков Ю.К., Краснова Н.К.** Аналитические структуры электрических обобщенно-однородных спектрографических сред // Научное приборостроение. 2014. Т. .24 1 №. С. 50—58.
19. **Голиков Ю.К., Краснова Н.К.** Обобщенный принцип подобия и его применение в электронной спектрографии // Прикладная физика. 2 № .2007. С. 11—5.
20. **Аверин И.А., Бердников А.С., Галль Н.Р.** Принцип подобия траекторий при движении заряженных частиц с разными массами в однородных по Эйлеру электрических и магнитных полях // Письма в Журнал технической физики. 2017. Т. 3 № .43. С. —39 43.
21. **Фихтенгольц Г.М.** Курс дифференциального и интегрального исчисления. Т. 1. М.: Физматлит, 616 .2001 с.
22. **Смирнов В.И.** Курс высшей математики. Т. 1. М.: Наука, 480 .1974 с.
23. **Бердников А.С., Аверин И.А., Краснова Н.К., Соловьёв К.В.** Об однородности скалярных и векторных потенциалов электрических и магнитных полей, однородных по Эйлеру // Успехи прикладной физики. 2017. Т. 1 № .5. С. 27—10.
24. **Гобсон Е.В.** Теория сферических и эллипсоидальных функций. М.: Изд-во иностр. лит-ры, 476 .1952 с.
25. **Бердников А.С., Краснова Н.К., Соловьёв К.В.** Теорема о дифференцировании трехмерных электрических и магнитных потенциалов, однородных по Эйлеру // Научное приборостроение. 2017. Т. 3 № .27. С. 107—119.
26. **Бердников А.С., Аверин И.А., Краснова Н.К., Соловьёв К.В.** Общие формулы для трехмерных электрических и магнитных потенциалов, однородных по Эйлеру, с целочисленным порядком однородности // Научное приборостроение. 2016. Т. 4 № .26. С. 13—30.
27. **Уиттекер Э.Т., Ватсон Дж.** Курс современного анализа. Ч. 2: Трансцендентные функции. М.: Гос. изд-во физ.-мат. лит-ры, 1963. 516 с.
28. **Donkin W.F.** On the equation of Laplace‘s functions &c. // Philosophical Transactions of the Royal Society of London. 1857. Vol. 147. Pp. 43—57.
29. **Donkin W.F.** On the equation of Laplace‘s functions &c. // Proceedings of the Royal Society of London. 1856 —1857. Vol. 8. Pp. 307—310.
30. **Голиков Ю.К.** Решение задачи Коши для однородных гармонических потенциалов нулевой кратности // Вестник Актюбинского регионального государственного университета им. К. Жубанова. Физико-математические науки. 44) 2 № .2016). С. 59—62.
31. **Голиков Ю.К., Бердников А.С., Антонов А.С., Краснова Н.К., Соловьёв К.В.** Применение формулы Донкина в теории электростатических призм // Журнал технической физики. 2018. Т. 11 № .88. С. 1711—1719.
32. **Гюнтер Н.М.** Интегрирование уравнений в частных производных первого порядка.

Ленинград, Москва: ОНТИ, 368 .1934 с.

33. **Трикоми Ф.** Лекции по уравнениям в частных производных. М.: Изд-во иностр. лит-ры, 443 .1957 с.

34. **Рашевский П.К.** Геометрическая теория уравнений с частными производными. Москва, Ленинград: ГИТТЛ, 362 .1947 с.

35. **Фиников С.П.** Метод внешних форм Картана в дифференциальной геометрии. Москва, Ленинград: ОГИЗ, 432 .1948 с.

36. **Картан Э.** Внешние дифференциальные системы и их геометрические приложения. М.: Изд-во Московского ун-та, 237 .1962 с.

37. **Щербаков Р.Н.** Основы метода внешних форм и линейчатой дифференциальной геометрии. Томск: Изд-во ТГУ, 236 .1973 с.

38. **Капцов О.В**. Методы интегрирования уравнений с частными производными. М.: Физматлит, 184 .2009 с.

39. **Янушаускас А.** Гармонические по М.А. Лаврентьеву отображения // Partial Differential Equations. Banach Center Publications. 1983. Vol. 10. No. 1. Pp. 213–241.

40. **Янушаускас А.И.** Трехмерные аналоги конформных отображений. Новосибирск: Наука, 173 .1982 с.

41. **Коппенфельс В., Штальман Ф.** Практика конформных отображений. М.: Изд-во иностр. лит-ры, 406 .1963 с.

42. **Лаврентьев М.А., Шабат Б.В.** Методы теории функций комплексного переменного. М.: Наука, 716 .1965 с.

43. **Маркушевич А.И.** Теория аналитических функций. В 2 тт. М.: Наука, 1968. Т. 1 – 486 с., Т. 2 – 624 с.

44. **Евграфов М.А.** Аналитические функции. -3е изд., перераб. и доп. М.: Наука, 448 .1991 с.

45. **Гурвиц А., Курант Р.** Теория функций. М.: Наука, 646 .1968 с.

46. **Клейн Ф.** Высшая геометрия. Москва, Ленинград: ГОНТИ, 400 .1939 с.

47. **Решетняк Ю.Г.** Теорема Лиувилля о конформных отображениях при минимальных предположениях регулярности // Сибирский математический журнал. 1967. Т. 8. № 4. С. 835–840.

48. **Flanders H.** Liouville’s theorem on conformal mapping // Journal of Mathematics and Mechanics. 1966. Vol. 15. No. 1. Pp. 157–161.

49. **Phillips R.** Liouville’s theorem // Pacific Journal. 1969. Vol. 28. No. 2. Pp. 397–405.

50. **Blair D.E.** Inversion theory and conformal mapping. Ser. “Student Mathematical Library”. Vol. 9 // American Mathematical Society. 2000. 118 p.

51. “Liouville’s theorem (conformal_mappings)” // URL: https://en.wikipedia. org/wiki/.

52. **Рашевский П.К.** Введение в Риманову геометрию и тензорный анализ. Москва, Ленинград: ОНТИ НКТП СССР, 1936. 199 с.

53. **Kobayashi S.** Transformation groups in differential geometry. Ser. “Classics in Mathematics”. Vol. 70. Berlin, New York: Springer-Verlag, 1995. 182 p.

54. **Wilker J.B.** Inversive geometry // Davis C., Grünbaum B., Sherk F.A. (Eds.) The Geometric Vein. New York: Springer, 1981. Pp. 379–442.

55. “Möbius transformation”. “Higher dimensions” // URL: https://en. wikipedia.org/wiki/

56. **Корн Г., Корн Т.** Справочник по математике для научных работников и инженеров. Определения, теоремы, формулы. Пер. со 2-го американского переработанного издания. М.: Наука, 1973. 831 с.

57. **Буляница А.Л., Курочкин В.Е.** Исследование процессов упорядочивания в открытых системах // Научное приборостроение. 2000. Т. 10. № 2. С. 43–49.

58. **Буляница А.Л., Евстрапов А.А., Рудницкая Г.Е.** Метод моментов при расчете параметров каналов в микроразмерных системах // Научное приборостроение. 2003. Т. 13. № 4. С. 28–40.

59. **Евстрапов А.А., Буляница А.Л., Рудницкая Г.Е., Беленький Б.Г., Петряков А.О., Курочкин В.Е.** Особенности применения алгоритмов цифровой фильтрации электрофореграмм при анализе веществ на микрочипе // Научное приборостроение. 2003. Т. 13. № 2. С. 57–63.

60. **Буляница А.Л.** Математическое моделирование в микрофлюидике: основные положения // Научное приборостроение. 2005. Т. 15. № 2. С. 51–66.

61. **Brzezina M., Šimůnková M.** On the harmonic morphism for the Kolmogorov type operators // Král J., Lukeš J., Netuka I., Veselý J. (Eds.) Potential Theory – ICPT-94. Berlin, New York: Walter de Gruyter, 1996. Pp. 341 –358.

62. Bibliothèque nationale de France // URL: http://gallica.bnf.fr.

63. Proceedings of the Royal Society of London // URL: http://rspl.royalsocietypublishing.org.

64. Numdam, the French digital mathematics library // URL: http://www.numdam.org/.

65. **Floquet G.** Sur les equations différentielles linéaires à coefficients périodiques // Annales scientifiques de l'École Normale Supérieure. 2e Serié. 1883. Vol. 12. Pp. 47–88.

66. HathiTrust Digital Library // URL:

https://www.hathitrust.org/.
67. Das Göttinger Digitalisierungszentrum (GDZ) // URL: https://gdz.sub.uni-goettingen.de/.
68. Wolfram mathematica // URL : http://wolfram.com/mathematica/

Статья поступила в редакцию 21.03.2019, принята к публикации 15.04.2019.

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