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SEMI-BOUNDED STRING'S VIBRATIONS INITIATED BY THE BOUNDARY REGIME

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Transverse vibrations of a semi-bounded string consisting of different materials are considered. The homogeneous wave equation with piecewise constant coefficients stand duty as a mathematical model. As a first step, we have investigated the solution of this equation with zero Cauchy data. The existence and uniqueness of the generalized solution of the problem were proved and the properties of the solution were analyzed. The specificity of the obtained conclusions was discussed, in particular, the zones of oscillations' propagation and of their absence were demonstrated. The obtained results are of a constructive character and can serve as a basis for the creation of a numerical algorithm. The importance of such problems is caused by their use in the theory of sensing inhomogeneous media by physical signals.

Keywords: differential equation, discontinuous coefficient, sounding of unknown media, wave process

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КОЛЕБАНИЯ ПОЛУОГРАНИЧЕННОЙ СТРУНЫ, ИНИЦИИРОВАННЫЕ ГРАНИЧНЫМ РЕЖИМОМ

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Рассматриваются поперечные колебания полуограниченной струны, состоящей из различных материалов. Математической моделью служит однородное волновое уравнение с кусочно-постоянными коэффициентами. В качестве первого этапа исследуется решение этого уравнения с нулевыми данными Коши. Доказывается существование и единственность обобщенного решения поставленной задачи, и анализируются его свойства. Отмечается специфичность полученных выводов, в частности, указываются зоны распространения колебаний и их отсутствия. Полученные результаты имеют конструктивный характер и могут служить основой создания численного алгоритма. Актуальность подобных задач вызвана их использованием в теории зондирования неоднородных сред физическими сигналами.

Ключевые слова: дифференциальное уравнение, разрывный коэффициент, зондирование неизвестных сред, волновой процесс

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Introduction

In this study, we have considered solutions of differential equations with discontinuous coefficients for higher derivatives. This field of research is still in its infancy, with no consistent results available. All the same, a number of studies of this type have been carried out [1–13]; the findings in [8–10] are perhaps closest to our own.

The essence of this problem is as follows. Considering a plane of variables (x,t) in the first quadrant

$$\mathbb{R}_2^{++}=((x,t),\ x>0,\ t>0)$$

we take the equation

$$\alpha(x)\frac{\partial^2 u(x,t)}{\partial t^2}-\beta(x)\frac{\partial^2 u(x,t)}{\partial x^2}=0, \quad (x,t)\in\mathbb{R}_2^{++},\ \alpha(x),\beta(x)>0 \tag{1}$$

and additional conditions

$$u(0,t)=\mu(t),\ u(x,0)=0,\ u_t(x,0)=0 \tag{2}$$

It is assumed that the function $\mu(t)$ has continuous derivatives up to and including the second order and the consistency conditions are satisfied:

$$\mu(0)=\mu'(0)=\mu''(0)=0, \tag{3}$$

These conditions coincide with the traditional requirements formulated in monograph [11].

For convenience, we assume that the function $\mu(t)$ is extended by zero at $t < 0$. Problem (1), (2) consists in finding the function $u(x,t)$ with the given functions $\alpha(x)$, $\beta(x)$, $\mu(t)$. Equations for solving this problem with constant α, β are well known and are given, for example, in monograph [11].

The case we are considering has not been studied previously: it is when the functions $\alpha(x)$, $\beta(x)$ are piecewise-constant:

$$\alpha(x)=\alpha_1,\ 0\le x\le x_0,\ \alpha(x)=\alpha_2,\ x>x_0;$$

$$\beta(x)=\beta_1,\ 0\le x\le x_0,\ \beta(x)=\beta_2,\ x>x_0,$$

where x_0, α_1, α_2, β_1, β_2 are positive constant numbers.

Equalities (1), (2), in particular, are a mathematical model for the process of transverse vibrations of a semi-bounded string. The vibrations are caused in this case only by the behavior of the boundary point ($x = 0$). According to our initial assessment, studying a more general problem with a nonzero right-hand side of Eq. (1) and nonzero Cauchy data $(u(x,0), u_t(x,0))$ would be incredibly cumbersome; we plan to tackle this task step-by-step in the future.

We should also note that the given problem (1), (2) is rather peculiar both because a relatively simple method can be used to solve it and because of the conclusions. In particular, we are going to find the zone of propagation of vibrations and the zone with no vibrations. The structure of these zones depends on the values of discontinuous coefficients of Eq. (1) and differs from the classical case.

Notations and definitions adopted

Aside from traditional notations, we are also going to use $\partial_1\chi(x,t)$, $\partial_2\chi(x,t)$ for the first derivatives of an arbitrary function χ(x,t), differentiable with respect to x and t.

Let us introduce the following unit vectors with characteristic directions:

$$\omega_1^+=\left(\frac{\sqrt{\beta_1}}{\sqrt{\alpha_1+\beta_1}},\frac{\sqrt{\alpha_1}}{\alpha_1+\beta_1}\right),$$

$$\omega_1^-=\left(\frac{-\sqrt{\beta_1}}{\sqrt{\alpha_1+\beta_1}},\frac{\sqrt{\alpha_1}}{\sqrt{\alpha_1+\beta_1}}\right),$$

$$\omega_2^+=\left(\frac{\sqrt{\beta_2}}{\sqrt{\alpha_2+\beta_2}},\frac{\sqrt{\alpha_2}}{\sqrt{\alpha_2+\beta_2}}\right),$$

$$\omega_2^-=\left(\frac{-\sqrt{\beta_2}}{\sqrt{\alpha_2+\beta_2}},\frac{\sqrt{\alpha_2}}{\sqrt{\alpha_2+\beta_2}}\right).$$

Let us denote

$$\gamma_1=\sqrt{\alpha_1\beta_1},\gamma_2=\sqrt{\alpha_2\beta_2},$$

$$a_1=\sqrt{\beta_1}/\sqrt{\alpha_1},$$

$$a_2=\sqrt{\beta_2}/\sqrt{\alpha_2}.$$

The following sets are taken in the quadrant $\mathbb{R}_2^{++}$:

$$G_1=\{(x,\ t):\ 0<x<x_0,\ t>0\}$$

$$G_2=\{(x,\ t):\ x>x_0,\ t>0$$

$$G_0=G_1\cup G_2$$

Line integrals of the second kind are widely used in our study. The notation (PQ) is used for a curve starting at point $P = (p_1, p_2)$ and ending at point $Q=(q_1, q_2)$. If the curve is the boundary of a simply-connected bounded domain, the orientation adopted is such that the domain is located on the left for a point moving along the curve. We have assumed that the points P and Q belong to the curve (PQ).

A generalized solution is sought in the class of functions $u(x,t)$ satisfying conditions (2) and Eq. (1) in the domains G_1 and G_2. In general, $u(x,t)$ is assumed to be continuous for $x \ge 0$, $t \ge 0$ and have partial derivatives in the domains G_1 and G_2, which are uniformly continuous in any bounded subdomain in G_1 and G_2. It is additionally assumed that the below coupling conditions are satisfied on the ray (x_0,t), $t > 0$:

$$\lim_{x\to x_0-0}\partial_2 u(x,t)=\lim_{x\to x_0+0}\partial_2 u(x,t), \qquad (4)$$

$$\lim_{x\to x_0-0}\beta_1\partial_1 u(x,t)=\lim_{x\to x_0+0}\beta_2\partial_1 u(x,t), \qquad (5)$$

which are consequences of Hooke's law and the law of conservation of momentum.

Monograph [11] considered a problem with the same sense as the problem (1), (2) that we have formulated, but under different restrictions, namely, with an integro-differential equation with respect to the function $u(x,t)$ studied instead of Eq. (1):

$$J(G)=\int_{\partial G}\alpha(\xi)\partial_2 u(\xi,\tau)d\xi+ +\beta(\xi)\partial_1 u(\xi,\tau)d\tau=0. \qquad (6)$$

The argument G in equality (6) is an arbitrary simply-connected domain in $\overline{\mathbb{R}_2^{++}}$ and its boundary ∂G is a piecewise smooth line of class C^1. The function $u(x,t)$ is continuous in $\overline{\mathbb{R}_2^{++}}$ and its partial derivatives $\partial_1 u(x,t)$, $\partial_2 u(x,t)$ are piecewise continuous with possible discontinuities of the first kind on certain lines. In this case, discontinuities of $\partial_1 u(x,t)$, $\partial_2 u(x,t)$ are allowed within G and it is also possible that the line of discontinuities coincides with a part of ∂G. Then the derivatives $\partial_1 u(x,t)$, $\partial_2 u(x,t)$ are replaced in Eq. (6) by their limit values within domain G. Notably, Eq. (6) is a consequence of Hooke's law and the law of conservation of momentum. Accordingly, the conclusions obtained from equality (6) also follow from these laws.

Let us call Eq. (6) with condition (2) problem (6), (2). Notice that within the problem statement formulated in [11], the presence of variables $\alpha(x)$, $\beta(x)$ is allowed in Eq. (6); however, all conclusions are drawn for constant coefficients.

Construction of composite characteristics emanating from a point in domain G_1

From now on, we are going to repeatedly use the following simple statement for the functions $u(x,t)$ described in problem (6), (2):

Lemma. *The following equalities hold true:*

$$\int_{(PQ)}\alpha(\xi)\partial_2 u(\xi,\tau)d\xi+\beta(\xi)\partial_1 u(\xi,\tau)d\tau= =\gamma_i(u(Q)-u(P)), \qquad (7)$$
$$(PQ)\subset\overline{G_i},(PQ)=\left\{P-\tau\omega_i^+,\tau\in[0,|Q-P|]\right\},$$
$$i=1,2;$$

$$\int_{(PQ)}\alpha(\xi)\partial_2 u(\xi,\tau)d\xi+\beta(\xi)\partial_1 u(\xi,\tau)d\tau= =-\gamma_i(u(Q)-u(P)), \qquad (8)$$
$$(PQ)\subset\overline{G_i},(PQ)= =\left\{P+\tau\omega_i^-,\tau\in[0,|Q-P|]\right\},$$
$$i=1,2.$$

P r o o f. Using the following representation in the left-hand side of the formula (7):

$$(PQ)=\left(p_1-s\frac{\sqrt{\beta_i}}{\sqrt{\alpha_i+\beta_i}},p_2-s\frac{\sqrt{\alpha_i}}{\sqrt{\alpha_i+\beta_i}}\right),$$
$$0\le s\le|Q-P|,$$

we can then proceed to an ordinary definite integral:

$$\int_{(PQ)}\alpha_i\partial_2 u(\xi,\tau)d\xi+\beta_i\partial_1 u(\xi,\tau)d\tau=$$
$$=\int_0^{|QP|}\left[\alpha_i\partial_2 u\left(p_1-s\frac{\sqrt{\beta_i}}{\sqrt{\alpha_i+\beta_i}},\right.\right.$$
$$\left.p_2-s\frac{\sqrt{\alpha_i}}{\sqrt{\alpha_i+\beta_i}}\right)\frac{-\sqrt{\beta_i}}{\sqrt{\alpha_i+\beta_i}}+$$
$$+\beta_i\partial_1 u\left(p_1-s\frac{\sqrt{\beta_i}}{\sqrt{\alpha_i+\beta_i}},\right.$$
$$\left.\left.p_2-s\frac{\sqrt{\alpha_i}}{\sqrt{\alpha_i+\beta_i}}\right)\frac{-\sqrt{\alpha_i}}{\sqrt{\alpha_i+\beta_i}}\right]ds=$$
$$=\sqrt{\alpha_i\beta_i}\int_0^{|QP|}\frac{d}{ds}\left[u(P-s\omega_i^+)\right]=$$
$$\gamma_i(u(Q)-u(P)).$$

As a result, the equalities obtained prove formula (7); formula (8) is proved in a similar fashion.

The lemma is proved.

Next, let us construct the following scheme.

Let us draw a segment of the straight line $x=a_1t$ from the origin until it intersects the line $x=x_0$ at point $P=(x_0,p)$. We then draw a ray lying on line $x_0=a_2(t-p)$ and located in the domain G from point P. The domain bounded from above by the segment and the ray and from below by the semi-axis Ox, $x > 0$ is denoted by G_3.

Let us prove that equality (6) implies another equality:

$$u(x,t)=0,\ (x,t)\in G_3$$

We take an arbitrary point $H=(x_0,h)$ in domain G_3.

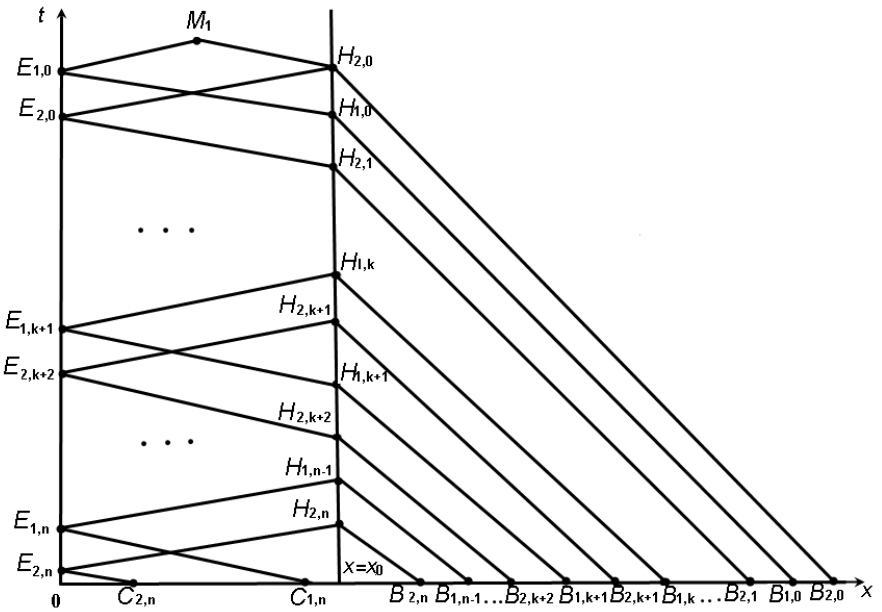

Fig. 1. Illustration to constructing two composite characteristics emanating from point $M_1=(x,t)$, $0 < x < x_0$, $t > a_1x$ on graph of $x(t)$

The straight lines

$$x-x_0=a_1(t-h),\ x-x_0=-a_2(t-h)$$

intersect the semi-axis Ox, $x > 0$ at points C and B, respectively.

We take a triangle $G(H)$ with vertices at points H, B, C as domain G; based on the lemma and formula (6) we obtain:

$$\int_{(PQ)} \alpha_i\,\partial_2 u(\xi,\tau)d\xi+\beta_i\partial_1 u(\xi,\tau)d\tau=$$

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Then, in view of equalities

$$u(C) = u(B) = 0$$

we obtain the equality $u(H) = 0$.

Now we take the point

$$M = (x,t) \in G_1 \cap G_3$$

The line $\xi - x = a_1(\tau-t)$ passing through point M intersects the semi-axis Ox, $x>0$ at point C'. Let us now consider a line

$$\xi - x = -a_1(\tau-t)$$

also passing through point M, taking the more complicated of the two possible options when this line crosses the ray (x_0,t), $t > 0$, at point $H=(x_0,h)$ in the domain G_3. Next, we draw through point H a straight line

$$\xi - x_0 = - a_2(\tau-h),$$

intersecting the semi-axis Ox, $x > 0$ at point B'.

Let us consider a quadrangle $G(M)$ with vertices M, C', B', H. We apply formula (6) to the domain $G=G(M)$ and calculate the integrals along the straight sections of its boundary; in view of the equalities

$$u(C') = u(B') = u(H) = 0,$$

we obtain the equality $u(M)=0$.

The reasoning is completely the same for point $M = (x,t) \in G_2 \cap G_3$.

It is known from the theory of differential equations that the characteristics for Eq. (1) are segments of lines

$$x=\pm a_1 t+const$$

in the domain G_1, and segments of lines

$$x=\pm\ a_2 t+const$$

in the domain G_2.

We take the point $M_1 \in G_1 \backslash \overline{G}_3$, i.e.,

$$M_1 = (x,\ t),\ 0 < x < x_0,\ t > a_1 x$$

(Fig. 1). We draw two continuous composite characteristics lying in the domain $\overline{G}_1$ from point M; segments of these characteristics passing through point $P = (p_1, p_2)$ are parts of straight lines

$$\xi - p_1 = \pm a_1(\tau - p_2).$$

These segments have endpoints on the rays.

$$R_1 = (0,t),\ t > 0,$$

$$R_2 = (x_0,t),\ t > 0$$

The points lying on the ray R_1 are denoted as

$$E_{i,k} = (0, e_{i,k}), i = 1,2, k = 0,1,...,$$

and the points on the ray R_2 as $H_{i,k}=(x_0,h_{i,k})$.

In this case, the index i numbers the composite characteristics, and the index k the endpoints.

The first composite characteristic is obtained by the following rule. A straight line

$$\xi - x = a_1(\tau - t)$$

is drawn from point M_1 until it intersects the ray R_1 at point $E_{1,0} = (0,\ e_{1,0})$. Next, a straight line

$$\xi = - a_1(\tau - e_{1,0})$$

is drawn through the obtained point $E_{1,0}$ until it intersects the ray R_2 at point $H_{1,0} = (x_0,h_{1,0})$. Further construction consists in drawing the lines

$$\xi = - a_1(\tau - e_{1,k})$$

through the obtained points $E_{1,k} = (0,\ e_{1,k})$ on ray R_1, intersecting ray R_2 at points $H_{1,k} = (x_0,\ h_{1,k})$. Straight lines

$$\xi - x_0 = a_1(\tau - h_{1,k})$$

are drawn through points $H_{1,k}$.

The formulae for these points have the following form:

$$e_{1,k} = t - \frac{x}{a_1} - 2k\frac{x_0}{a_1},$$

$$h_{1,k} = t - \frac{x}{a_1} - (2k+1)\frac{x_0}{a_1},\ k \geq 0. \tag{9}$$

Construction continues until the obtained segments of the characteristics have a non-empty intersection with the domain G_1. Thus, we have obtained a set of points $E_{1,k}$, $H_{1,k}$ on rays R_1, R_2 respectively, $0 \leq k \leq n$. For final construction, we use the straight line

$$\xi = - a_1(\tau - e_{1,n}),$$

intersecting the semi-axis Ox, $x > 0$, at point $C_{1,n}$.

The second composite characteristic emanating from point M_1 is constructed similarly, i.e., a straight line $\xi - x_0 = - a_1(\tau - t)$ is drawn from point M_1 until it intersects ray R_2 at point $H_{2,0}$. The sequence of steps we take then is alternately using the characteristics $\xi = \pm a_1\tau + const$ emanating from the points already obtained; we get a set of endpoints $E_{2,k}$, $0 \leq k \leq n$, on ray R_1 and $H_{2,k}$, $0 \leq k \leq n$, on ray R_2 and also point $C_{2,n}$ on the horizontal axis $(0,\ x_0)$.

The formulae for these points have the following form:

$$e_{2,k} = t - \frac{x_0 - x}{a_1} - (2k+1)\frac{x_0}{a_1},$$

$$h_{2,k} = t - \frac{x_0 - x}{a_1} - 2k\frac{x_0}{a_1}, \tag{10}$$

$$k \geq 0,\ C_{2,n} = (0, c_{2,n}).$$

Construction of the characteristics used is simpler in the set $\overline{G}_2$: straight lines

$$\xi - x_0 = - a_2(\tau - h_{i,k})$$

are drawn through points $H_{i,k}$, $i = 1, 2$, $k = 0,...,n$, until they intersect the semi-axis Ox, $x > 0$ at points $B_{i,k}$.

Consequences from Eq. (6) for the function $u(x,t)$ in domains G_1, G_2

Let us consider a polygon $G(M_1)$ with vertices at points M_1, $E_{1,0}$, $H_{1,0}$, $B_{1,0}$, $B_{2,0}$, $H_{2,0}$ and apply formula (6) to it for the case $G = G(M_1)$ (see Fig. 1).

Since

$$\partial G(M_1) = (M_1E_{1,0}) \cup (E_{1,0}H_{1,0}) \cup (H_{1,0}B_{1,0}) \cup$$
$$\cup (B_{1,0}B_{2,0}) \cup (B_{2,0}H_{2,0}) \cup (H_{2,0}M_1),$$

we calculate the integral $J(G(M_1))$ using the lemma

$$J(G(M_1)) = \gamma_1(u(E_{1,0}) - u(M_1)) -$$
$$-\gamma_1(u(H_{1,0}) - u(E_{1,0})) - \gamma_2(u(B_{1,0}) -$$
$$-u(H_{1,0})) - \gamma_2(u(H_{2,0}) - u(B_{2,0})) -$$
$$-\gamma_1(u(M_1) - u(H_{2,0})) = 0.$$

It follows then, in view of equalities $u(B_{1,0}) = u(B_{2,0}) = 0$, that

$$u(M_1) = u(E_{1,0}) + \frac{\gamma_2 - \gamma_1}{2\gamma_1} \times$$
$$\times u(H_{1,0}) + \frac{\gamma_1 - \gamma_2}{2\gamma_1} u(H_{2,0}). \tag{11}$$

To find the expression for $u(H_{2,0})$, let us consider a polygon $G(H_{2,0})$ with vertices at points $H_{2,0}$, $E_{2,0}$, $H_{2,1}$, $B_{2,1}$, $B_{2,0}$. Similar to the calculations we have just performed, using the lemma for the case $G = G(H_{2,0})$, we obtain the following:

$$u(H_{2,0}) = \frac{2\gamma_1}{\gamma_2 + \gamma_1} u(E_{2,0}) + \frac{\gamma_2 - \gamma_1}{\gamma_2 + \gamma_1} u(H_{2,1}).$$

To find the expression for $u(H_{2,1})$, let us consider a polygon $G = G(H_{2,1})$ with vertices

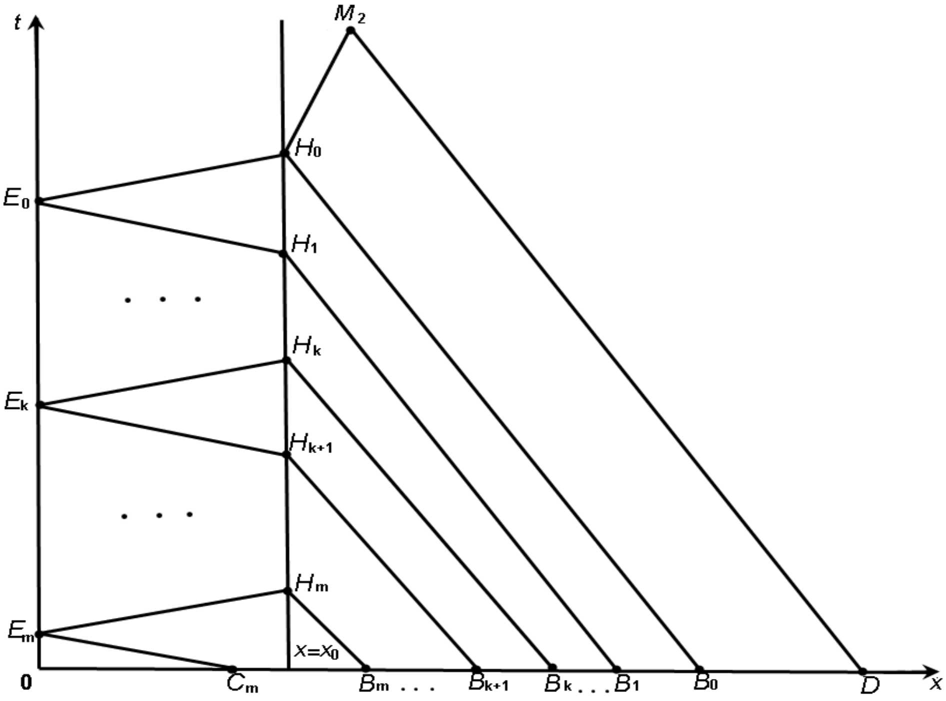

Fig. 2. Illustration to constructing two composite characteristics emanating from point $M_2 \in G_2\backslash\overline{G}_3$ on graph of $x(t)$

at points $H_{2,1}$, $E_{2,1}$, $H_{2,2}$, $B_{2,2}$, $B_{2,1}$. Using the lemma again and repeating the calculations, we obtain the formula

$$u(H_{2,1})=\frac{2\gamma_1}{\gamma_2+\gamma_1}u(E_{2,1})+\frac{\gamma_2-\gamma_1}{\gamma_2+\gamma_1}u(H_{2,2}).$$

In the same way, we can obtain the following general formula:

$$u(H_{2,k})=\frac{2\gamma_1}{\gamma_2+\gamma_1}u(E_{2,k})+ +\frac{\gamma_2-\gamma_1}{\gamma_2+\gamma_1}u(H_{2,k+1}). \quad (12)$$

Using the resulting recurrent formula (12), we obtain:

$$u(H_{1,0})=\frac{2\gamma_1}{\gamma_2+\gamma_1}\sum_{k=0}^{\infty}\left(\frac{\gamma_2-\gamma_1}{\gamma_2+\gamma_1}\right)^k u(E_{1,k+1}). \quad (13)$$

To find the formulae for $u(H_{1,0})$ in a similar manner but using the elements of the second composite characteristic emanating from point M_1, we obtain the following equality:

$$u(H_{1,0})=\frac{2\gamma_1}{\gamma_2+\gamma_1}\sum_{k=0}^{\infty}\left(\frac{\gamma_2-\gamma_1}{\gamma_2+\gamma_1}\right)^k u(E_{1,k+1}). \quad (14)$$

Notably, only a finite number of first terms is non-zero in the right-hand sides of equalities (13) and (14), due to condition (3), $\mu(t)=0$, $t \le 0$. Using Eqs. (13), (14) for formula (11), we end up with

$$u(M_1)=u(E_{1,0})+\sum_{k=0}^{\infty}\left(\frac{\gamma_2-\gamma_1}{\gamma_2+\gamma_1}\right)^{k+1}\times \times\left(u(E_{1,k+1})-u(E_{2,k})\right). \quad (15)$$

Let us now consider a simpler case when an arbitrary point $M_2=(x,t)$ belongs to the domain $G_2\backslash\overline{G}_3$. We construct a continuous composite characteristic emanating from this point and ending on the horizontal axis. For this purpose, we once again use the segments of the characteristics lying on the straight lines

$$\xi=\pm a2\ \tau+\text{const}.$$

We denote the endpoints of the individual characteristics obtained on the ray $x=x_0$, $t>0$, by H_k, $k=0, 1, ..., m$ and those on the ray $x=0$, $t>0$ by E_k, $k=0, 1..., m$. We denote the endpoints of the individual characteristics on the horizontal axis in ascending order by C_m, B_m, B_{m-1},..., B_0, D (Fig. 2).

By similar reasoning as in the previous case, we obtain the following formulae:

$$u(M_2)=\frac{2\gamma_1}{\gamma_2+\gamma_1}\sum_{k=0}^{\infty}\left(\frac{\gamma_2-\gamma_1}{\gamma_2+\gamma_1}\right)^k u(E_k), \quad (16)$$

$$E_k=\left(0, t-\frac{x}{a_2}+\left(\frac{1}{a_2}-\frac{1}{a_1}\right)x_0-2k\frac{x_0}{a_1}\right). \quad (17)$$

Notably, the situation with constructing composite characteristics (see Figs. 1 and 2) does not cover all possible cases, for example, when the last segment of the composite characteristic emanates from point $H_{2,n}$ or H_m and ends on the horizontal axis. We have also analyzed this case, establishing that it does not change the formulae obtained.

Main results

We are going to prove in this section that the formulae obtained above allow solving problem (1), (2).

Theorem. There is a unique solution *u(x,t)* of problem (1), (2), represented by two equalities: (15) for all $M_1 \in G_1$ and (16) for any point $M_2 \in G_2$.

Proof. Let us carry out the proof in two stages.

1. Existence of the solution. The function $u(x,t)$, $(x,t) \in G_0 \backslash \overline{G}_3$, represented by formulae (15), (16), satisfying Eq. (1) directly follows from the fact that each term in the corresponding series is a solution of this equation. As for set $\overline{G}_3$, it was proved that $u(x,t) = 0$, $(x,t) \in \overline{G}_3$. The remaining task is then to verify whether the rest of the properties we have described in setting the problem are satisfied.

First, notice that if the point M_1 tends to line $\xi = a_1\tau$, then, as follows from formulae (9), (10),

$$\mu(e_{1,k}) \to 0,\ \mu(e_{2,k}) \to 0,$$

which means that $u(x,t)$, $(x,t) \in G_1$ is continuous.

It follows from the same considerations that the first and second-order partial derivatives of the function $u(x,t)$, $(x,t) \in G_1$ are continuous. By similar reasoning, we can easily prove that the requirement for the function $u(x,t)$, $(x,t) \in G_2$ to be smooth is satisfied. It follows from the above that formulae (15), (16) give functions satisfying Eq. (1) everywhere in domain G_0.

Next, we note that that if point $M_1 = (x,t)$ tends to point $(0,t)$, then

$$E_{1,k} \to \left(0, t - 2k\frac{x_0}{a_1}\right),$$

$$E_{2,k} \to \left(0, t - (2k+2)\frac{x_0}{a_1}\right),$$

$$E_{1,0} \to (0,t).$$

It follows then that $|E_{2,k} - E_{1,k+1}| \to 0$. Therefore, the right-hand side of equality (15) tends to $\mu(t)$, which means that the boundary condition $u(0,t)=\mu(t)$ is satisfied. Thus, we proved that conditions (2) are satisfied for the function $u(x,t)$.

Now we have to verify whether the properties of *u(x,t)* are satisfied with $x \to x_0$. Evidently, the following relation holds true then:

$$E_k, E_{1,k}, E_{2,k} \to \left(0, t - (2k+1)\frac{x_0}{a_1}\right).$$

Consequently, if $x \to x_0$, formula (15) takes the form

$$\begin{aligned} u(x_0,t) &= u(E_0) + \sum_{k=0}^{\infty}\left(\frac{\gamma_2-\gamma_1}{\gamma_2+\gamma_1}\right)^{k+1} \times \\ &\times\big(u(E_{k+1}) - u(E_k)\big) = \frac{2\gamma_1}{\gamma_2+\gamma_1}u(E_0) + \\ &+\sum_{k=1}^{\infty}\left(\left(\frac{\gamma_2-\gamma_1}{\gamma_2+\gamma_1}\right)^{k} - \left(\frac{\gamma_2-\gamma_1}{\gamma_2+\gamma_1}\right)^{k+1}\right)u(E_k) = \\ &= \frac{2\gamma_1}{\gamma_2+\gamma_1}\sum_{k=0}^{\infty}\left(\frac{\gamma_2-\gamma_1}{\gamma_2+\gamma_1}\right)^{k}u(E_k). \end{aligned} \tag{18}$$

The right-hand side of the obtained equality (18) coincides with the right-hand side of equality (16), which proves that the function $u(x,t)$ is continuous for $x \to x_0$.

Let us now prove that conditions (4), (5) are fulfilled. It follows from equality (15) that

$$\frac{\partial u(M_1)}{\partial t} = \mu'(e_{1,0}) + \sum_{k=0}^{\infty}\left(\frac{\gamma_2-\gamma_1}{\gamma_2+\gamma_1}\right)^{k+1} \times \big(\mu(e_{1,k+1}) - \mu(e_{2,k})\big).$$

From here we obtain:

$$\begin{aligned} \lim_{x\to x_0-0}\frac{\partial u(M_1)}{\partial t} &= \mu'\left(t - \frac{x_0}{a_1}\right) + \\ &+\sum_{k=1}^{\infty}\left(\frac{\gamma_2-\gamma_1}{\gamma_2+\gamma_1}\right)^{k}\left(\mu'\left(t-(2k+1)\frac{x_0}{a_1}\right) - \right. \\ &\left. -\mu'\left(t-(2k-1)\frac{x_0}{a_1}\right)\right). \end{aligned} \tag{19}$$

Next, it follows from equality (16) that

$$\begin{aligned} \lim_{x\to x_0+0}\frac{\partial u(M_2)}{\partial t} &= \frac{2\gamma_1}{\gamma_2+\gamma_1}\sum_{k=0}^{\infty}\left(\frac{\gamma_2-\gamma_1}{\gamma_2+\gamma_1}\right)^{k}\mu' \times \\ &\times\left(t-(2k+1)\frac{x_0}{a_1}\right). \end{aligned} \tag{20}$$

Following the steps similar to those taken to obtain equality (18) for equality (19), we get:

$$\lim_{x\to x_0+0}\frac{\partial u(M_2)}{\partial t} = \lim_{x\to x_0-0}\frac{\partial u(M_1)}{\partial t};$$

and the resulting equality means that condition (4) is fulfilled.

Practically the same steps are taken to verify whether condition (5) is fulfilled:

$$\lim_{x\to x_0-0}\frac{\partial u(M_1)}{\partial x}=-\frac{1}{a_1}\frac{2\gamma_2}{\gamma_2+\gamma_1}\times$$

$$\times\sum_{k=0}^{\infty}\left(\frac{\gamma_2-\gamma_1}{\gamma_2+\gamma_1}\right)^k\mu'\left(t-(2k+1)\frac{x_0}{a_1}\right);$$

$$\lim_{x\to x_0+0}\frac{\partial u(M_2)}{\partial x}=-\frac{1}{a_2}\frac{2\gamma_1}{\gamma_2+\gamma_1}\times$$

$$\times\sum_{k=0}^{\infty}\left(\frac{\gamma_2-\gamma_1}{\gamma_2+\gamma_1}\right)^k\mu'\left(t-(2k+1)\frac{x_0}{a_1}\right).$$

It follows from the last two equations that

$$\beta_2\lim_{x\to x_0+0}\frac{\partial u(M_2)}{\partial x}=\beta_1\lim_{x\to x_0-0}\frac{\partial u(M_1)}{\partial x},$$

and this means that condition (5) is satisfied.

Thus, we have proved that the solution exists.

2. *Uniqueness of the solution*. To prove this, we take two solutions of problem (1), (2) and denote their difference as $V(x,t)$.

We consider the functions

$$\nu_1(x,t)=\partial_2 V(x,t)+a(x)\partial_1 V(x,t),$$

$$\nu_2(x,t)=\partial_2 V(x,t)-a(x)\partial_1 V(x,t).$$

It is easy to verify that the following equalities hold true:

$$\partial_2\nu_1(x,t)-a(x)\,\partial_1\nu_1(x,t)=0,$$
$$\partial_2\nu_2(x,t)+a(x)\,\partial_1\nu_2(x,t)=0, \quad (21)$$

$$\nu_i(0,t)=0,\ \nu_i(x,0)=0,$$
$$i=1,2,\ (x,t)\in G_0. \quad (22)$$

Let us agree to denote $\nu_1(x,t)$, $\nu_2(x,t)$, $V(x,t)$ in terms of

$$v_1^-(x,t),\ v_2^-(x,t),V^-(x,t),$$

for $0 < x < x_0$, and for $x \geq x_0$

$$v_1^+(x,t),v_2^+(x,t),V^+(x,t).$$

Therefore, we obtain the following equation from Eqs. (21) and (22):

$$v_1^-(x,t)=v_2^-(x,t)=0.$$

From here and from conditions (4), (5) the equalities

$$v_1^+(H)=v_2^+(H)=0$$

follow for an arbitrary point H on the ray (x_0,t), $t > 0$.

Then, it follows from these equalities and equalities (21), taking into account conditions

$$v_1^+(x,0)=v_2^+(x,0)=0,$$

that

$$v_1^+(x,t)=v_2^+(x,t)=0.$$

Thus, we obtain the equalities

$$v_1(x,t)=v_2(x,t)=0,$$

$$\partial_1 V(x,t)=0,\partial_2 V(x,t)=0,$$

$$V(x,t)=\text{const}.$$

Therefore, by virtue of the condition $V(x,t)=0$, we obtain:

$$V(x,t)\ =\ 0,\ (x,t)\in\overline{\mathbb{R}_2^{++}},$$

which actually means that the solution of the problem is unique.

The theorem is proved.

Conclusion

We have considered a one-dimensional wave equation describing not only the transverse vibrations of an inhomogeneous semi-bounded string but also longitudinal vibrations of an inhomogeneous rod. We have posed a problem of finding the vibration function for a particular case when the process is caused solely by the behavior of the boundary point.

We have proved the theorem that the solution for this problem exists and is unique, and provided simple and explicit formulae for this solution. A compact form for writing the solution is given for the theorem, using convenient auxiliary notations.

More complete formulas containing only the initial data of the problem have the following form:

$$(x,t)\in G_1,\ u(x,t)=\mu\left(t-\frac{x}{a_1}\right)+$$

$$+\sum_{k=0}^{\infty}\left(\frac{\gamma_2-\gamma_1}{\gamma_2+\gamma_1}\right)^{k+1}\left(\mu\left(t-\frac{x}{a_1}-2(k+1)\frac{x_0}{a_1}\right)-\right.$$

$$\left.-\mu\left(t-\frac{x}{a_1}-(2k+1)\frac{x_0}{a_1}\right)\right),$$

$$(x,t)\in G_2,\ u(x,t)=\frac{2\gamma_1}{\gamma_2+\gamma_1}\sum_{k=0}^{\infty}\left(\frac{\gamma_2-\gamma_1}{\gamma_2+\gamma_1}\right)^k\times$$

$$\times\mu\left(t-\frac{x}{a_2}+\left(\frac{1}{a_2}-\frac{1}{a_1}\right)x_0-2k\frac{x_0}{a_1}\right).$$

Importantly, the last of the given formulae allow to easily construct the corresponding numerical algorithm.

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