SHARP V-NOTCH FRACTURE CRITERIA UNDER ANTIPLANE DEFORMATION

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The criteria for brittle fracture of a sharp V-notch when it is loaded with antiplane concentrated forces have been considered: a criterion for the maximum average stress, a criterion for the average energy density of deformation, and an approach based on the joint use of the force and energy criteria. Failure loads estimates on the basis of the exact solutions and using asymptotics of stresses near the V-notch tip were found. A comparative analysis of the failure loads obtained through those criteria was carried out. For the asymmetric loading, the initial angle of the crack propagation from the V-notch tip was determined. In the calculation of this angle, the application of the stress asymptotics was shown to result in significant errors and to require the consideration of regular terms in the stress representations.

Key words: antiplane deformation, sharp V-notch, fracture criterion, average stress, deformation energy

Citation: V.V. Tikhomirov, Sharp V-notch fracture criteria under antiplane deformation, St. Petersburg Polytechnical State University Journal. Physics and Mathematics. 11 (3) (2018) 77–83. DOI: 10.18721/JPM.11309

Introduction

The tIps of sharp v-notches in elastic bodies are singular points of stress fields. Cracks may develop from these singularities under certain conditions, leading to fracture of elastic structures. For this reason, study of the stress-strain state of bodies with notches, as well as developing fracture criteria and confirming them experimentally are issues of particular interest.

Plane cracks with known fracture criteria (first developed by Griffith and Irwin) are a particular case of notches to which these criteria are not directly applicable. In view of this, several other criteria for fracture of structural elements with sharp notches have been proposed:

force [1 – 5];

energy [6 – 10];

formulated within the framework of so-called finite fracture mechanics and based on combined application of force and energy conditions [11, 12].

The majority of studies applied these criteria within the framework of a plane problem for structures with finite or semi-infinite notches. The criteria were in fact used based on asymptotic representations of stresses near the stress concentrations. It was established that the critical values of fracture parameters, such as ultimate loads, can be expressed in terms of macroscopic characteristics of materials, such as ultimate tensile strength and fracture toughness.

Some recent studies estimated the effect that including non-singular terms in stress expansions in the vicinity of the notch tip has on parameters such as the generalized stress intensity factor [13] and the crack initiation angle [14]. The results of these studies indicate that including a non-singular first term in the Williams expansion significantly affects these fracture parameters.

Antiplane problems related to our subject matter were discussed in very few studies [8, 15, 16], with no comparative analysis of fracture criteria for wedge-shaped structures.

The main goal of this study is to extend the fracture criteria developed for plane problems to the case of antiplane deformation of notched bodies and comparative analysis of these criteria in determining ultimate loads.

Since an elastic solution for a uniform wedge-shaped region can be obtained in closed form as explicit representations for stresses and displacements in case of antiplane loading, it is possible to estimate the accuracy of calculating the fracture load using stress asymptotes at the tip of the notch.

Green's functions for a sharp notch

Let us consider antiplane deformation of a homogeneous isotropic wedge-shaped region with a vertex angle 2α $(\pi/2 < \alpha \le \pi)$. The notch then is defined by the angle $\beta \in [0, \pi)$. A concentrated force $2T$ directed from the plane is applied to the face of the wedge $\theta = \alpha$ at a distance r_0 from the tip. The general problem on finding the stress-strain state of a plane with a notch is linear, so it can be represented as a superposition of two problems:

1) with symmetric loading of the notch faces, when

$$\tau_{\theta z}(r, \alpha) = \tau_{\theta z}(r, -\alpha) = T\delta(r - r_0), \quad (1)$$

2) with antisymmetric loading, when

$$\tau_{\theta z}(r, \alpha) = -\tau_{\theta z}(r, -\alpha) = T\delta(r - r_0). \quad (2)$$

($\delta(r - r_0)$ is the Dirac delta function).

Next, let us apply a Mellin integral transform to the harmonic equilibrium equation, satisfying boundary conditions (1) and (2); in addition, let us use the residue theorem. As a result, we obtain the following representations for stresses:

in problem 1:

$$\tau_{\theta z}(r, \theta) = \frac{K_3^N}{\sqrt{2\pi} r^{1-\lambda}} \frac{1 + \rho^2}{1 + 2\rho^2 \cos 2\lambda\theta + \rho^4} \cos \lambda\theta,$$
$$\tau_{rz}(r, \theta) = \frac{K_3^N}{\sqrt{2\pi} r^{1-\lambda}} \frac{1 - \rho^2}{1 + 2\rho^2 \cos 2\lambda\theta + \rho^4} \sin \lambda\theta; \quad (3)$$

in problem 2:

$$\tau_{\theta z}(r, \theta) = \frac{K_3^N}{\sqrt{2\pi} r^{1-\lambda}} \frac{\rho}{1 + 2\rho^2 \cos 2\lambda\theta + \rho^4} \sin 2\lambda\theta;$$
$$\tau_{rz}(r, \theta) = -\frac{K_3^N}{\sqrt{2\pi} r^{1-\lambda}} \frac{\rho(\rho^2 + \cos 2\lambda\theta)}{1 + 2\rho^2 \cos 2\lambda\theta + \rho^4}. \quad (4)$$

$$\rho = (r/r_0)^\lambda.$$

Here, $\lambda = \pi/(2\alpha)$ and $\lambda = 1$ in the case of a half-plane $(\alpha = \pi/2)$ and $\lambda = 1/2$ in the case of a semi-infinite crack in an unbounded plane $(\alpha = \pi)$.

The quantity K_3^N in relations (3) is the generalized stress intensity factor (GSIF), defined by the formula

$$K_3^N = \lim_{r \to 0} \sqrt{2\pi} r^{1-\lambda} \tau_{\theta z}(r, 0) = \frac{\sqrt{2\pi} T}{\alpha r_0^\lambda}. \quad (5)$$

With $\alpha = \pi$, the GSIF coincides with the stress intensity factor (SIF) at the tip of a semi-infinite crack:

$$K_3^N(\pi) = K_3 = T\sqrt{\frac{2}{\pi r_0}}. \quad (6)$$

When concentrated forces take critical values equal to T_c, formulae (5) and (6) define the critical intensity factors.

$$K_{3c}^N = \frac{\sqrt{2\pi} T_c}{\alpha r_0^\lambda}, \quad K_{3c} = T_c\sqrt{\frac{2}{\pi r_0}}. \quad (7)$$

It should be emphasized that, in contrast with the fracture toughness constant, the critical stress intensity factor K_{3c}^N at the tip of the notch is not a constant of the material, since it depends on the angle α. We should also note that stresses (3) at the tip of the notch in problem 1 have a power singularity, while stresses (4) in problem 2 do not. Stress asymptotes (3) with $r \to 0$ are determined by the formulae

$$\tau_{\theta z}(r, \theta) = \frac{K_3^N}{\sqrt{2\pi}} r^{1-\lambda} \cos \lambda\theta,$$
$$\tau_{rz}(r, \theta) = \frac{K_3^N}{\sqrt{2\pi}} r^{1-\lambda} \sin \lambda\theta. \quad (8)$$

Notably, formulae (3) for stresses are consistent with the results given in [17].

Summing solutions (3) and (4), we obtain the stresses in the problem on the action of a concentrated force $2T$ at the face of the notch $\theta = \alpha$:

$$\tau_{\theta z}(r, \theta) = \frac{K_3^N}{\sqrt{2\pi}} r^{1-\lambda} \frac{\cos \lambda\theta}{1 - 2\rho \ \sin \ \lambda\theta + \rho^2},$$
$$\tau_{rz}(r, \theta) = \frac{K_3^N}{\sqrt{2\pi}} r^{1-\lambda} \frac{\sin \lambda\theta - \rho}{1 - 2\rho \ \sin \ \lambda\theta + \rho^2}. \quad (9)$$

Evidently, stresses (9) have asymptotes (8) if $r \to 0$.

Criteria for sharp notch fracture

Let us consider application of the fracture criteria with the example of a notch with symmetrically loaded faces. In this case, by virtue of symmetry, the stress $\tau_{\theta z}$ reaches the maximum value on the ray $\theta = 0$ and, therefore, the crack is going to propagate from the tip of the notch along this ray.

Force criterion. Similarly to the assumptions adopted in [1, 2], we assume that fracture of the notch starts when the maximum mean stress calculated at a certain distance *d* from its tip reaches a critical value equal to the shear strength τ_c of the material:

$$\bar{\tau} = \frac{1}{d}\int_0^d \max_{-\alpha<\theta<\alpha} \tau_{\theta z}(r,\theta)dr = \tau_c. \quad (10)$$

Substituting the stress $\tau_{\theta z}$ found by formula (3) with $\theta = 0$ to expression (10), we obtain the following equality:

$$\frac{K_{3c}^N(\alpha) r_0^\lambda}{\sqrt{2\pi\lambda d}} \operatorname{arctg}\left(\frac{d}{r_0}\right)^\lambda = \tau_c. \quad (11)$$

It is valid for any value of the angle $\alpha \in (\pi/2, \pi]$.. Let us determine the parameter *d* for the angle $\alpha = \pi$, i.e., for the case of a crack with $\lambda = 1/2$ and $K_{3c}^N(\pi) = K_{3c}$, in other words, mode III fracture toughness. Then we obtain the following equation for determining the relative distance $x = d/r_0$ from condition (11):

$$x = \gamma \operatorname{arctg}\sqrt{x}. \quad (12)$$

Here we have introduced a dimensionless parameter

$$\gamma = \sqrt{\frac{2}{\pi r_0}} \frac{K_{3c}}{\tau_c}, \quad (13)$$

whose equivalent was used for plane problem in [12], where a different linear dimension, notch depth, was used instead of the distance r_0 to the load application point. This parameter was called the brittleness parameter, or the brittleness number in [12].

Let us estimate the value of γ using the example of a brittle material such as graphite. The fracture toughness of graphite in mode III is, according to [18], K_{3c} = 0.415 MPa·m$^{1/2}$. Since $\tau_c = \sqrt{3}\sigma_c$ (σ_c is the ultimate tensile strength, which takes the value of 20 MPa for graphite [19]), we obtain, according to formula (13), $\gamma = 0{,}0287/\sqrt{r_0}$. Then, for example, if r_0 = 0.01 m, we obtain the value $\gamma = 0{,}287$.

Using criterial relation (11) and representation (7), we obtain the estimate for the ratio of critical forces for the case of a notch and a crack:

$$\frac{T_c^N}{T_c} = \frac{x}{\gamma \operatorname{arctg}(x^\lambda)}. \quad (14)$$

With $\gamma << 1$ the root of equation (12) can be represented as

$$x = \gamma^2 + O(\gamma^3)$$

and, therefore, the asymptote of the relative critical load (14) is determined by the formula

$$\frac{T_c^N}{T_c} = \gamma^{1-2\lambda}. \quad (15)$$

Since λ lies in the range $1/2 \le \lambda < 1$ for any value of the angle α from $\pi/2 < \alpha \le \pi$, the inequalities $-1 < 1 - 2\lambda \le 0$ hold true. Then it follows from formula (15) that larger forces have to be applied for fracture of a sharp notch at small values of the parameter γ, compared to the forces required for crack propagation. In other words, a crack, considered as a limiting case of a notch with $\alpha \to \pi$, can be regarded as the most dangerous notch. This conclusion agrees qualitatively with the result obtained for uniaxial tension of the notch in [12].

Notably, if only the singular terms of stress expansion (3), that is, asymptotes (8), are used in fracture criterion (10), then we also obtain equality (15) for the ultimate load. Thus, the estimates of the fracture load, constructed from exact and asymptotic solutions, coincide if the distances $\mathbf{r}_0$ from the tip of the notch to the force application points are large enough.

Energy criterion. Fracture of the notch by a forming crack starts when the mean deformation energy density, calculated in a finite volume of radius *R* with the center at the tip of the notch, reaches a critical value Π_c [6]:

$$\frac{1}{2\mu\alpha R^2}\int_0^R\int_{-\alpha}^{\alpha} (\tau_{rz}^2 + \tau_{\theta z}^2) r dr d\theta = \Pi_c, \quad (16)$$

where μ is the shear modulus of the material.

The radius of the control volume R depends on the properties of the material.

The critical value of the mean deformation energy density assuming that it does not depend on the vertex angle of the notch can be expressed in terms of the shear strength of the material τ_c :

$$\Pi_c = \tau_c^2/(2\mu).$$

Then, using the representations for stresses (3) for the critical state of the material and calculating the integrals in criterion (16), we obtain the equality

$$\frac{(K_{3c}^N)^2 r_0^{2\lambda}}{8\lambda^2 \alpha R^2} \ln \frac{1+(R/r_0)^{2\lambda}}{1-(R/r_0)^{2\lambda}} = \tau_c^2. \quad (17)$$

In the limiting case, when the notch degenerates into a crack, i.e., with $\alpha = \pi$ and, consequently, with $K_{3c}^N(\pi) = K_{3c}$, equality (17) yields the equation for determining the radius of the control volume:

$$y = \frac{\gamma}{2}\sqrt{\ln\frac{1+y}{1-y}}, \quad y = \frac{R}{r_0}. \quad (18)$$

In view of equality (7) and equation (18), condition (17) leads to the following estimate of the fracture load for the notch:

$$\frac{T_c^N}{T_c} = \sqrt{\frac{2}{\lambda}}\frac{y}{\gamma}\Bigg/\sqrt{\ln\frac{1+y^{2\lambda}}{1-y^{2\lambda}}}. \quad (19)$$

We obtain from formula (19) with $y = R/r_0 << 1$, that

$$\frac{T_c^N}{T_c} = \frac{1}{f(\lambda)}\gamma^{1-2\lambda}, \quad (20)$$

and the function $f(\lambda) = 2^{1-\lambda}\sqrt{\lambda} \ge 1$ for any $\lambda \in [0,5;\ 1,0]$.

Then, comparing estimates (15) and (20), we conclude that the ultimate load obtained from the force criterion exceeds the ultimate load found using the mean energy density criterion at any angle $\alpha \in (\pi/2, \pi]$.

Notably, using energy criterion (16) with only asymptotic representations (8) also leads to an estimate of the form (20).

Criterion based on finite fracture mechanics [12]. Criterion based on finite fracture mechanics [12]. In this case, it is assumed that two conditions must be simultaneously satisfied for finite propagation Δ of a crack from the top of the notch: the force and the energy conditions (for stresses and energy balance):

$$\int_0^\Delta \tau_{\theta z}(r,0)dr \ge \tau_c\Delta, \qquad \int_0^\Delta K_3^2(\varepsilon)d\varepsilon \ge K_{3c}^2\Delta, \quad (21)$$

where $K_3(\varepsilon)$ is the stress intensity factor (SIF) at the tip of the crack of length ε.

Thus, in order to use this criterion, we need to obtain, in addition to stress field (3), the solution of the problem on a crack of finite length ε, propagating from the tip of the notch (see Fig. 1).

Let us now apply the Mellin integral transform to the harmonic equilibrium equation, to condition (1) on the face $\theta = \alpha$ and to the following mixed conditions on the beam $\theta = 0$:

$$\tau_{\theta z}(r,+0) = 0 \quad (0 \le r \le \varepsilon),$$

$$w(r,+0) = 0 \quad (\varepsilon \le r < \infty).$$

As a result, we obtain the Wiener – Hopf equation:

$$\mathrm{ctg}(p\alpha)\ T_-(p) + \frac{\mu}{\varepsilon}U_+(p) = \frac{Tr_0^p}{\varepsilon^{p+1}\sin(p\alpha)} \quad (p \in L). \quad (22)$$

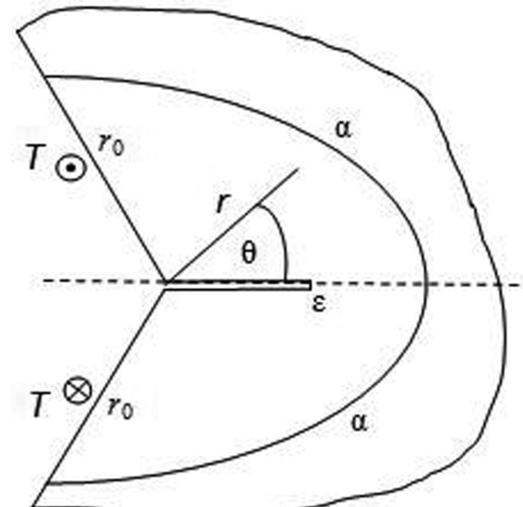

Fig. 1. A sharp notch with a symmetric crack of length ε propagating from its tip; 2α is the vertex angle of the wedge-shaped region; r_0 is the distance from the tip to the application point of concentrated forces T, directed from the plane; r, θ are the coordinates

Here p is the Mellin transform parameter. The stress transforms $T_-(p)$ and displacement transforms $U_+(p)$ along the beam are analytical functions in the left and right (relative to the contour L) half-planes.

Using the technique developed in [20], we obtain the exact solution of equation (22), which allows to express the SIF at the crack tip as

$$K_3 = K_3^N \psi(\lambda)\varepsilon^{\lambda-1/2}, \qquad (23)$$

where $\psi(\lambda) = \{2\lambda[1+(\varepsilon/r_0)^{2\lambda}]\}^{-1/2}$.

Substituting stresses (3) and SIF (23) into criterion (21) (at the critical state of the notch), we obtain the equalities

$$K_{3c}^N \frac{r_0^\lambda}{\lambda\sqrt{2\pi}} \operatorname{arctg}(\Delta/r_0)^\lambda = \tau_c \Delta,$$
$$(K_{3c}^N)^2 \frac{r_0^{2\lambda}}{4\lambda^2} \ln[1+(\Delta/r_0)^{2\lambda}] = K_{3c}^2 \Delta. \qquad (24)$$

From here we obtain the equation determining relative propagation of a crack with $\varsigma = \Delta/r_0$:

$$\varsigma = \gamma^2 \frac{\operatorname{arctg}^2 \varsigma^\lambda}{\ln(1+\varsigma^{2\lambda})}. \qquad (25)$$

Using equalities (7), we find from the first equation in (24) the relative fracture load in the form

$$\frac{T_c^N}{T_c} = \frac{\varsigma}{\gamma \operatorname{arctg} \varsigma^\lambda}. \qquad (26)$$

Notice that equation (25) has a root $\varsigma \approx \gamma^2$ with $\varsigma \ll 1$. In this case, equality (26) leads to the following estimate of the fracture load:

$$\frac{T_c^N}{T_c} = \gamma^{1-2\lambda},$$

which coincides with formula (15) for using the force criterion of fracture.

The angle of initial propagation of the crack under asymmetric loading of the notch

To determine the initial angle of crack propagation from the tip of the notch under asymmetric loading, let us use, for example, the force criterion proposed within the framework of the plane problem [5]. Crack initialization occurs along the beam $\theta = \theta_*$, where the mean shear stress takes the maximum value:

$$\bar{\tau}_{\theta z}(\theta) = \frac{1}{d}\int_0^d \max_{-\alpha<\theta<\alpha} \tau_{\theta z}(r,\theta)dr, \qquad (27)$$

$$\left.\frac{\partial \bar{\tau}_{\theta z}(\theta)}{\partial \theta}\right|_{\theta=\theta_*} = 0. \qquad (28)$$

Substituting expression (9) into formula (27), we obtain the following representation for the mean tangential stress:

$$\bar{\tau}_{\theta z}(\theta) = \frac{K_3^N r_0^\lambda}{\sqrt{2\pi}\lambda d} \times \left[\operatorname{arctg}\frac{(d/r_0)^\lambda - \sin\lambda\theta}{\cos\lambda\theta} + \lambda\theta\right]. \qquad (29)$$

After using condition (28), we find from here the angle è$_*$ describing the direction of initial crack growth:

$$\theta_* = \frac{1}{\lambda}\arcsin(d/r_0)^\lambda. \qquad (30)$$

We should note that the follow estimate of the fracture load follows from fracture criterion (10) and formulae (29) and (30):

$$\frac{T_c^N}{T_c} = \frac{x}{\gamma \arcsin(x^\lambda)},$$

where $x = d/r_0$ is the root of the equation

$$x = \gamma \arcsin\sqrt{x}.$$

Numerical results and discussion

Based on the three given criteria, we have calculated fracture loads under symmetric loading of the notch depending on the parameter γ and different angles α. Comparative analysis of the results based on the exact solution of problem (3) shows that all the criteria yield similar results and the maximum discrepancy does not exceed 3 % for small values of the parameter ($\gamma < 0.1$). In this case, according to formulae (15) and (20), the fracture load has an asymptotic estimate $T_c^N/T_c = O(\gamma^{1-2\lambda})$.

With increasing parameter γ, the relative ultimate load decreases, and its values, determined using criteria (10), (16) and (21), diverge. The criterion based on finite fracture mechanics yields the greatest value for this load, and the criterion of mean deformation energy density provides a lower-bound estimate of the load. For example, with $\gamma = 0.8$ and a

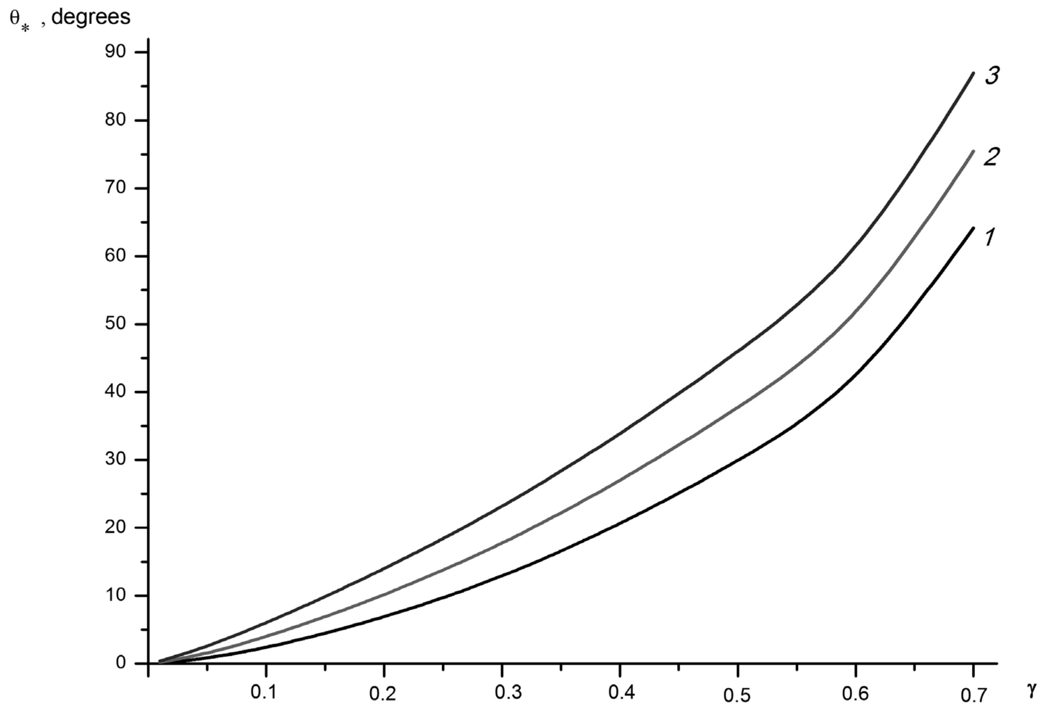

Fig. 2. Dependences of the initial angle of crack propagation from the tip of the notch on the parameter γ with different notch vertex angles α, deg: 120 (*1*) 135 (*2*), 150 (*3*)

notch with an angle of 90°, the difference in estimates for T_c^N/T_c based on these criteria is about 13 %.

The values of fracture loads for the notch found using stress asymptotes (8) almost coincide, up to $\gamma = 0.5$, with the values calculated from exact solution (3).

Thus, using the asymptotes of the stress field near the tip of the notch to estimate the fracture load within the framework of the antiplane problem is fairly acceptable.

Under asymmetric loading of the notch faces, the initial angle of crack propagation depends significantly on the regular terms in stress representation (9). Using only stress field asymptotes (9) in the form (8) with criterion (27), (28) determines the initial angle $\theta_*^{as} = 0$. However, this angle, calculated from exact solution (9) using formula (30), may considerably differ from the value of θ_*^{as} (Fig. 2). It follows then that non-singular terms have to be included in the formulae for stresses if $r \to 0$ for finding the direction of the initial growth of a crack from the tip of the notch.

Conclusion

The paper considers the criteria for brittle fracture of a sharp notch under antiplane loading with concentrated forces: a) maximum mean stress, b) mean deformation energy density, c) an approach based on combining force and energy criteria.

We have established that the fracture loads resulting from application of different criterial relationships are expressed in terms of a single dimensionless parameter depending on the material constants (shear strength and fracture toughness in mode III). Apparently, the ultimate loads found using different approaches are quite close.

However, the angle of initial propagation of a crack from the tip of the notch considerably depends on the accuracy of calculating the stresses near this tip, i.e, calculating this angle based on the stress asymptotes leads to significant errors.

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Received 19.03.2018, accepted 20.05.2018.

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