WEAK SOLUTIONS OF THE CROCCO BOUNDARY PROBLEMS

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A procedure for designing an approximate solution of the Crocco boundary typical problem has been proposed in the paper. The procedure calls for the change of this initial problem by a nonlinear integral equation. The latter was solved by direct calculation of the integral using the mean-value theorem. . The averaging parameter was eliminated by integrating over the parameter in the (0, 1) interval. Widening the scope of the solution procedure was demonstrated and weak solutions were found. For the classical case, the weak solution was not too different from the Blasius exact one. The approximate value of the Blasius constant turned out to be 1/3 and differed from the exact one (0.33206) by 0.3 %.

Key words: Cauchy problem, integral equation, mean value theorem, group of transformations, solitary wave

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Introduction

Crocco's boundary problems are primarily used for hydrodynamic applications, in particular, longitudinal viscous flow past a plate, with unsteady seepage in homogeneous and isotropic (scalar) porous media [1 – 3].

A typical Crocco boundary problem is stated as follows [1]:

$$\begin{gathered} 2\varphi \frac{d^2\varphi}{du^2} + u = 0, \\ D(\varphi) = (u : 0 \le u_0 < u < 1), \varphi \in C^{(2)}(D(\varphi)), \\ \left(\frac{d\varphi}{du}\right)_{u=u_0} = \varphi(1) = 0, \\ \mathrm{Im}(\varphi) = (0, a), a := \varphi(u_0), \end{gathered} \tag{1}$$

and $u_0 = 0$ in the classical Blasius case.

Problem (1) in the given form is widely used in hydrodynamics, where the variable u is interpreted as the longitudinal velocity and the distribution φ as the shear stress [1].

Problem (1) is involved in seepage theory for calculating a solitary flow-rate wave, i.e., for solving the boundary problem for the Boussinesq equation [2, 3]:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial s}\left(ku\frac{\partial u}{\partial s}\right), \tag{1a}$$

where $u = u(t,s) \le 1$ is the seepage flow depth ($t > 0$, $s > 0$); k is the hydraulic conductivity.

In a particular case,
$u(0,s) - 1 = u(t,0) = 0$.

In the general case,

$$k = k(u),\ c = -k(u)\left(\frac{\partial u}{\partial s}\right)^c,$$

where c is the seepage rate.

In the classical Boussinesq case, $k = 1$, $c = 1$. The Boussinesq equation then takes the form

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial s}\left(u\frac{\partial u}{\partial s}\right). \tag{1b}$$

Finally, the Crocco equation is used in problems on jet motion of viscous fluid (free convection in heated channels, free submerged and near-wall jets, etc.) [4, 5].

In contrast to the "natural" statement of applied boundary problems, problem (1) is convenient because it allows finding an injective mapping of the compact set $(u_0, 1)$ into a compact set

$$(0, a), \varphi : (u_0, 1) \to (0, a).$$

More specifically, we suppose that each branch of the solution of boundary problem (1) is a 2-diffeomorphism $\varphi : (u_0, 1) \to (0, a)$.

The solutions of boundary problem (1) are given in [6 – 32]. These studies fall into two classes:

the first class uses analytical methods, including solutions obtained in the form of power and splitting (flat) series;

studies of the second class primarily rely on numerical solutions.

Analytical papers (belonging to the first class) include those using methods of the theory of Lie transformation groups and expansions into power series and enveloping series.

For example, Boussinesq equation (1b) admits a linear transformation

$$z = \alpha t \pm \sqrt{\alpha} s,$$

and, consequently, there exists a solution of the Boussinesq equation in the form of a solitary flow-rate wave:

$$z = u + c_1 - c_2 \ln(u + c_2).$$

The case $c_2 = 0$ in this solution corresponds to a centered flow-rate wave propagating with a velocity of $\pm\alpha^{1/2}$ either upstream or downstream.

A particular group of studies [26 – 31] offer analytical tools for constructing solutions of ordinary differential equations (ODEs) of the given type near a singular point. These studies emphasize that the properties of an analytic function mainly depend on its singularities, which cannot be examined in the real interval. Moving onto the complex plane automatically means constructing a mapping onto the Riemann surface of the solution [31].

"Exact" solutions of the boundary problem for the Crocco equation are also obtained by using power series with respect to *u*. However, the Tauberian theorems are unknown for these solutions: namely, the series for the function $\varphi(u)$ turn out to be "poorly" convergent for $u \to 1-0$. For example, the divergence of the series for $\varphi(u)$ at the outer edge of the boundary layer $(u \to 1-0)$ and bifurcation of the solution in the outer (i.e., jet) part of the boundary layer was established in [32].

We have omitted to discuss the so-called integral methods, where, instead of the equations for the distribution density, ODEs are solved for the actual distributions (integral relations), in the above overview. These methods are ideologically closer to completely different types of methods such as direct or variational.

Thus, flat series for analytical solutions are currently the only alternative to numerical methods for solving Blasius, Chazy, and Crocco equations.

The goal of this study has been to construct an approximate solution of a typical Crocco boundary problem using the averaging procedure.

Constructing the solution of the problem

In this study, we use the method of constructing an approximate solution of boundary problem (1), based on replacing this initial boundary problem by an integral equation, and subsequently introducing an artificial parameter and averaging over this parameter. In other words, a distribution (functional) with a density coinciding with the approximate solution is used instead of an exact solution.

This technique consists in the following. Let

$$f(u) \in C^{(1)}(0,1),\ f(u) \geq 0.$$

The Crocco boundary problem for the interval (0, 1) has the form

$$2\varphi \frac{d^2\varphi}{du^2} + f(u) = 0, \varphi'(0) = \varphi(1) = 0$$

and admits a formal order reduction:

$$2\frac{d\varphi}{du} = -\int_0^u \frac{f(v)dv}{\varphi(v)}.$$

For a positive branch of the solution $\varphi := \varphi^+$, the function $1/\varphi(u)$ is a positive monotonically decreasing distribution mapping the interval $u \in (u_0,\ 1)$ onto the interval $\varphi \in (0, a)$, where $a := \varphi(0)$.

Then, according to the Bonnet theorem,

$$\frac{d\varphi_\theta^2}{du} = -\int_{\theta u}^{u} f(v)dv, \qquad (\#)$$

where $0 < \theta < 1$ is the parameter (proper fraction).

Let $u = 1$ in this formula. In this case, with $u \to 1 - 0$,

$$d\varphi_\theta^{\ 2} / du = O(1).$$

This means that the equalities

$$\varphi_\theta(u) = O(\varepsilon^m), \ d\varphi_\theta / du = O(\varepsilon^{-m}),$$
$$m = n,$$

where m, n are positive parameters, have to hold true.

Integrating equality (#) one more time, we obtain:

$$\varphi_\theta^2(u) = \int_u^1 dv \int_{\theta v}^{v} f(t)dt.$$

This equality is actually the approximate θ solution of the Crocco equation (indicated by the subscript θ). This solution depends continuously on the fraction (parameter) θ, and, evidently,

$$\varphi_1^2(u) = 0, \varphi_0^2(u) = \int_u^1 dv \int_0^u f(t)dt =$$
$$= \int_0^1 (1-t)f(t)dt - \int_0^u (u-t)f(t)dt > \varphi_\theta^2(u) > 0,$$
$$\forall \theta \in (0,1),$$

while, generally speaking, $\partial\varphi / \partial\theta < 0$.

The parameter θ can be eliminated, for example, by averaging the derivative with respect to it:

$$\frac{d\varphi^2}{du} := \int_0^1 \frac{d\varphi_\theta^2}{du} d\theta,$$

which leads to the expression

$$\frac{d\varphi^2}{du} = -1 / u \int_0^u vf(v)dv.$$

Finally, we can write the following approximate solution of boundary problem (1):

$$\varphi^2(u) = \int_0^1 vf(v) \ln\frac{1}{v} dv - \int_0^u vf(v) \ln\frac{u}{v} dv. \quad (2)$$

The result obtained does not depend on the order of integration with respect to the parameter θ and to the argument u. Let us call solution (2) a weak θ solution.

Properties of solutions of boundary problem (1)

We are going to list the properties of the solutions of boundary problem (1) in this section (the proofs of these properties are omitted).

1. The boundary conditions in problem (1) can be replaced by one-point (Cauchy) conditions:

$$\left(\frac{d\varphi}{du}\right)_{u=u_0} = \varphi(0) - a = 0, \quad (3)$$

with the parameter a in initial conditions (3) chosen so that $\varphi(1) = 0$. Imposing these conditions is justified by the continuous dependence of φ on the parameter a.

2. There are two branches of the solution of boundary problem (1) and, respectively, of one-point boundary problem (3):

$$\varphi^+(u) \text{ and } \varphi^-(u)$$

(Fig. 1). These branches are related as follows:

$$\varphi^+(u) + \varphi^-(u) = 0, \ 0 < u < 1,$$

and

$$0 \le \varphi^+(u) \le a, \frac{d\varphi^+}{du} < 0, \frac{d^2\varphi^+}{du^2} < 0;$$
$$a \le \varphi^-(u) \le 0, \frac{d\varphi^-}{du} > 0, \frac{d^2\varphi^-}{du^2} > 0.$$

Boundary problem (1) is typical, since, in particular, the homogeneous Crocco boundary problem is reduced to it. Let $u_0 = 0$, and then instead of representation (1) we consider the homogeneous Crocco boundary problem:

$$2\varphi\frac{d^2\varphi}{du^2} + u = 0, \quad (1c)$$
$$\varphi(0) = \varphi(1) = 0.$$

The solution of homogeneous boundary problem (1c) consists of two branches:

$$0 \le \varphi^+(u) \text{ and } \varphi^-(u) \le 0$$

(negative and positive), such that

$$\varphi^+(u) + \varphi^-(u) = 0$$

for each u value from the interval $0 < u < 1$.

There exists a value of $u = u^*$ $(0 < u^* < 1)$, such that $d\varphi^\pm / du^* = 0$ (according to Rolle's theorem). Therefore, each of the branches $\varphi^\pm(u)$ of the solution of homogeneous prob-

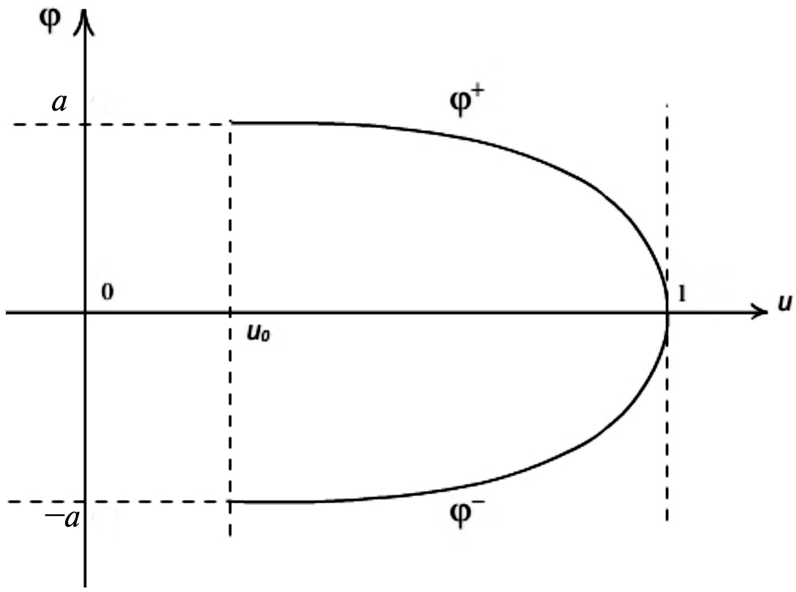

Fig. 1. Solution of a typical Crocco boundary problem: positive ($\varphi^+(u)$) and negative ($\varphi^-(u)$) monotonic branches; the vertical dashed lines indicate the boundaries of the interval

lem (1) is decomposed into two solutions of the typical Crocco boundary problem (1):

$$\varphi_l^{\pm}(u),\ D(\varphi_l^{\pm}) = (0, u^*);$$

$$\varphi_r^{\pm}(u),\ D(\varphi_r^{\pm}) = (u^*, 1).$$

In this case, the solutions $\varphi(u) = \varphi_{l,r}^{\pm}(u)$ of positive and negative typical boundary problems are mapped continuously and smoothly at the point $u = u^*$ (Fig. 2):

$$\varphi_l^{\pm}(0) = \left(\frac{d\varphi_l^{\pm}}{du}\right)_{u=u^*-0} = \left(\frac{d\varphi_r^{\pm}}{du}\right)_{u=u^*+0} = \varphi_r^{\pm}(1) = 0.$$

The branches of the solutions of the homogeneous boundary problem are decomposed into solutions of typical boundary problems

$$\varphi_l^{\pm}(u^* - 0) - \varphi_r^{\pm}(u^* + 0) = 0,$$
$$\left(\frac{d^2\varphi^+}{du^2}\right)_{u=u^*} < 0, \left(\frac{d^2\varphi^-}{du^2}\right)_{u=u^*} > 0. \qquad (\#\#)$$

3. Solution of boundary problem (1) – (3) satisfies the identity

$$\int_{u_0}^{1}\left(\frac{d\varphi}{du}\right)^2 du = \frac{1-u_0^2}{4}. \qquad (4)$$

4. Solution of boundary problem (1) – (3) is equivalent to the problem of the minimum positive functional (distribution):

$$F(\varphi) = (1/2)\int_{u_0}^{1}\left(\left(\frac{d\varphi}{du}\right)^2 + u\ln\frac{a}{\varphi}\right)du > 0.$$

In other words, the condition $dF \le \delta F$ is satisfied along the extremals of the functional $F(\varphi)$, where dF is the variation along the characteristic (the trajectory of the solution), and δF is the variation along the admissible (virtual) trajectory. The basis for the proof of property 4 is that the necessary minimum condition $F(\varphi)$ coincides with the Crocco equation, and the sufficiency of the condition is guaranteed by the convexity of the Lagrangian density $F(\varphi)$.

5. The property of the solution for $u_0 = 0$. In this case, boundary problem (1) is equivalent to the following nonlinear integral equation:

$$\frac{d\varphi}{du} = -\frac{1}{2}\int_0^u \frac{v\,dv}{\varphi(v)},$$

$$\varphi(u) = \frac{1}{2}\int_u^1 dv \int_0^v \frac{t\,dt}{\varphi(t)} = \frac{1}{2}\left(\int_0^1 \frac{(1-t)t\,dt}{\varphi(t)} - \int_0^u \frac{(u-t)t\,dt}{\varphi(t)}\right).$$

An iterative process can be used to solve this integral equation.

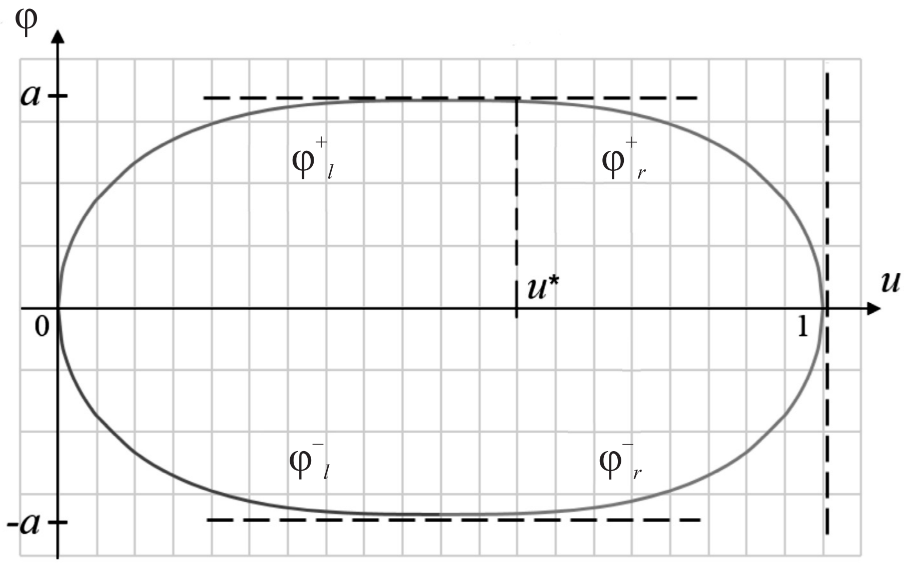

Fig. 2 . The $\varphi^{\pm}(u)$ dependences illustrating how the branches of the solution of a homogeneous boundary problem are decomposed into solutions (##) of typical boundary problems; u^* corresponds to the maxima of the function $|\varphi|$

Let the subscript denote the iteration number, and then the process of solution is expressed as

$$\frac{d\varphi_s}{du} = -\int_0^u \frac{vdv}{\varphi_{s-1}(v)},$$

$$\varphi_s(u) = \frac{1}{2}\int_u^1 dv \int_0^v \frac{tdt}{\varphi_{s-1}(t)}.$$

If $1/\varphi \in L_1(0,1)$, then $\varphi \in C^{(1)}(0,1)$. Assuming that the sequence of iterations of the function $1/\varphi_s$ forms a Cauchy sequence, $\varphi_s \to \varphi$ almost everywhere on the interval $0 < u < 1$, since the domain $L_1(0, 1)$ is complete.

6. The solution allows to formulate a corollary to the mean value theorem. Since $1/\varphi(u)$ is a monotonically increasing distribution, then, according to the Bonnet mean theorem, the equalities

$$2\varphi(u)\frac{d\varphi}{du} = -\frac{u^2}{2}(1-\theta^2),$$
$$\varphi_\theta^2(u) = \frac{1}{6}(1-u^3)(1-\theta^2), \qquad (1c)$$

where θ is a proper fraction ($0 < \theta < 1$), hold true.

The final expression for the approximate solution has the form

$$\varphi_\theta(u) = (1/\sqrt{6})\sqrt{(1-u^3)(1-\theta^2)}.$$

The mean square value of $\varphi_\theta(u)$, i.e., the θ approximation of the solution, is determined by the equality

$$\varphi^2(u) = \int_0^1 \varphi_\theta^2(u)d\theta = (1/9)(1-u^3). \qquad (5)$$

It follows from here that $\varphi(u) = (1/3)\sqrt{1-u^3}$, and this approximates the Blasius exact solution, especially for small u values. For example, the value of the Blasius constant is $a = 1/3$. Its exact value, recently calculated by Varin, is [29, 30]:

a = 0,33205733621519
629893718006201058
296654709356141267
981810047564019872
417401806440507049
0731855146368... .

The rational value of the constant differs from the reduced irrational one by less than 0.3 %.

The quantity $\mathfrak{D} := \int_0^1 \varphi du$ in problems with physical content is dissipation in the segment

(0, 1). In this case, its value is

$$\mathfrak{D} = (1/3)\int_0^1 \sqrt{1-u^3}\,du = \frac{\sqrt{\pi}}{18}\frac{\Gamma(1/3)}{\Gamma(11/6)} \approx 0,27.$$

Apparently, the distribution $\varphi(u)$ is responsible for dissipation in the neighborhood of the point $u = 0$.

7. Let us formulate the general definition of the norm of θ approximation

$$\varphi_r(u) = \frac{1}{\sqrt{6}}(1-u^3)\cdot\left(\int_0^1 (1-\theta^2)^{r/2}d\theta\right)^{1/r} =$$

$$= \frac{1-u^3}{\sqrt{6}}\left(\frac{\sqrt{\pi}\Gamma(r/2+1)}{2\Gamma(r/2+3/2)}\right)^{1/r},$$

where $r > 0$ is any positive real number.

We are going to omit the subscript from now on, and the norm should be clear from the context. For example, $r = 2$ was adopted in the previous subsection, and then we obtain that

$$\varphi(u) := \varphi_2(u) = \frac{1-u^3}{\sqrt{6}}\sqrt{\frac{\sqrt{\pi}}{2\Gamma(5/2)}} =$$

$$= \frac{1}{\sqrt{6}}\sqrt{\frac{2}{3}}(1-u^3) = (1/3)(1-u^3).$$

Next, the Cauchy – Hölder inequality

$$\varphi_r(u) \le \varphi_{r+\alpha}(u), \forall\alpha > 0,$$

holds uniformly with respect to $0 < u < 1$, and the sequence of norms does not decrease as the index r increases from 0 to ∞, or, more precisely,

$$(1/\sqrt{6})\exp(-c/2 - 0,02) < \|a\|_r < 1/\sqrt{6},$$

where c is the Mascheroni constant.

8. The first generalized property of the solution. Let the Crocco boundary problem (1) have the following form

$$2\varphi\frac{d^2\varphi}{du^2} + u^m = 0,$$
$$\varphi'(0) = \varphi(0) = 0.$$

This representation of the Crocco equation follows from the Boussinesq equation if

$$k = k(h) = k_0(h/H)^{m-1}.$$

The θ approximation of the solution of boundary problem (1a) then takes the form

$$\varphi_\theta^2(u) = \frac{(1-\theta^{b+1})(1-u^{m+2})}{(m+1)(m+2)}, \quad (5a)$$

and, if we apply mean-square averaging over θ, the weak solution has the form:

$$\varphi(u) = \frac{\sqrt{1-u^{m+2}}}{m+2}, \quad a = \frac{1}{m+2}. \quad (5b)$$

Identity (4) is written as follows:

$$\int_0^1 \left(\frac{d\varphi}{du}\right)^2 du = \frac{1}{2(m+1)}. \quad (4a)$$

The condition for a minimum quadratic functional practically does not change and is expressed as

$$F(\varphi) = (1/2)\int_0^1 \left(\left(\frac{d\varphi}{du}\right)^2 + u^m \ln\frac{a}{\varphi}\right)du \to \inf \ge 0.$$

In this case, dissipation follows the expression

$$\mathfrak{D} = \frac{1}{m+2}\int_0^1 \sqrt{1-u^{m+2}}\,du =$$

$$= \frac{1}{(m+2)^2}\frac{\Gamma\left(\frac{1}{m+2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3m+8}{2(m+2)}\right)} = \quad (6)$$

$$= O\left(\frac{1}{m+2}\right), m \gg 1,$$

and decreases with increasing m.

By virtue of identity (4a), the following expression holds true:

$$\bar{\varphi}(u) := \frac{\varphi(u)}{a} = \sqrt{1-u^{m+2}},$$

and it can be seen from this that the degree to which the profile

$$\bar{\varphi}(u) := \frac{\varphi(u)}{a}$$

is complete is increased with increasing parameter m.

Giving a physical meaning to the solution, let us assume that $\varphi = \varphi(u)$ is the friction in the boundary layer. The friction on the surface of a plate with longitudinal viscous flow over it is then $a = \varphi(0)$, decreases monotonically and persists from the near-wall to the jet part of the layer:

$$\text{with } m \to \infty, \varphi(u) \to \varphi(0) = 0,$$
$$0 < u < 1.$$

9. The second generalized property of the solution. Let the Boussinesq equation and the boundary conditions for it have the form

$$\frac{\partial u^a}{\partial t} = \frac{\partial}{\partial s}\left(u^b\left(\frac{\partial u}{\partial s}\right)^c\right),$$

where a, b, c are real parameters;

$$D(u) = (t > 0,\ s > 0),$$
$$u(0, s) - 1 = u(t, 0) = 0.$$

Then the corresponding Crocco transformation converts this equation to the form

$$u^b = \frac{a}{c+1}\varphi(u)\left(-\frac{d}{du}\left(u^{1-a}\frac{d\varphi}{du}\right)\right)^c, \quad (7)$$

and the boundary conditions are imposed as follows:

$$\varphi(1) = \varphi'(0) = 0. \quad (8)$$

In this case, the θ solution of boundary problem (7), (8) has the form

$$\varphi_\theta(u) = \frac{(c+1)c^{\frac{c}{c+1}}}{a^{\frac{1}{c+1}}(b+c)^{\frac{c}{c+1}}(b+ac)^{\frac{c}{c+1}}} \times$$
$$\times \{(1-\theta^{b/c+1})(1-u^{a+b/c+1})\}^{\frac{c}{c+1}}. \quad (9)$$

Note 1. If $a = b = 1$ in the particular case, the expression known as the Khristianianovich seepage model for flat seepage flow follows from (9):

$$\varphi_\theta(u) = \frac{\left(\left(1-\theta^{\frac{c+1}{c}}\right)\left(1-u^{\frac{2c+1}{c}}\right)\right)^{\frac{c}{c+1}}}{(c+1)^{\frac{c}{c+1}}}. \quad (9a)$$

Then

$$a_\theta = \left(\frac{1-\theta^{\frac{c+1}{c}}}{c+1}\right)^{\frac{c}{c+1}},$$

and, furthermore,

$$a^r = \frac{c}{(c+1)^{\frac{c(r+1)+1}{c}}}\,\frac{\Gamma\left(\frac{c}{c+1}\right)\cdot\Gamma\left(\frac{c(r+1)+1}{c+1}\right)}{\Gamma\left(\frac{c(r+2)+1}{c+1}\right)},$$

with $r \geq 0$.

Obviously,

$$\overline{\varphi}(u) := \frac{\varphi(u)}{a} = \left(1-u^{\frac{2c+1}{c}}\right)^{\frac{c}{c+1}},$$

and the profile of the dimensionless distribution $\overline{\varphi}(u)$ is completed with decreasing parameter c

For example, if $c = 1/2$, then

$$\overline{\varphi}(u) = (1-u^4)^{1/3};$$

and if $c = 1/3$, then

$$\overline{\varphi}(u) = (1-u^5)^{1/4}.$$

Note 2. Let $a = 1$, b, c be the free parameters. Then, by virtue of θ solution (9), we obtain the expressions

$$\varphi_\theta(u) = \frac{(c+1)c^{\frac{c}{c+1}}}{(b+c)^{\frac{2c}{c+1}}}\{(1-\theta^{b/c+1})(1-u^{b/c+2})\}^{\frac{c}{c+1}},$$

$$\overline{\varphi}(u) = (1-u^{b/c+2})^{\frac{c}{c+1}},$$

and the degree to which the profile $\overline{\varphi}(u)$ is completed is increased with increasing parameter b.

For example, if $c = 1/2$, then

$$\overline{\varphi}(u) = (1-u^{2(b+1)})^{1/3} \underset{b\to\infty}{\to} 0, \quad 0 < u < 1.$$

Conclusions

As a result of the study we have conducted, we have established the following:

A typical Crocco boundary problem admits a positive and a negative branch of the solution (φ^+ and φ^-), such that

$$\varphi^+(u) + \varphi^-(u) = 0.$$

A homogeneous Crocco boundary problem is reduced to two typical Crocco boundary problems, conjugate at the critical point $u = u^*$, such that

$$0 < u_0 < u^* < 1,$$

$$\left(\frac{d\varphi}{du}\right)_{u=u^*} = \varphi(u^* - 0) - \varphi(u^* + 0) = 0.$$

A typical Crocco boundary problem is equivalent to a nonlinear integral equation. The latter is solved by direct calculation of the integral using the second mean-value theorem. The averaging parameter is excluded by integration over the parameter in the interval (0, 1).

Extensions of the proposed solution method have been discussed in this paper. The weak θ solution is insignificantly different from the exact one for the classical case $a = b = c = 1$. The approximate value of the Blasius constant $a = \varphi(0)$ turns out to be 1/3. In this case, the exact value of $\varphi(0) = 0,33206$.

REFERENCES

[1] **L. Crocco,** Sulla strato limite laminare nei gas lungo una lamina plana, Rend. Math. Appl., Ser. 5. 21 (2) (1941) 138–152.

[2] **P.Ya. Polubarinova-Kochina,** Teoriya dvizheniya gruntovykh vod [Groundwater movement theory], 2nd ed., Nauka, Moscow, 1977.

[3] **M.R. Petrichenko,** Predelnaya zadacha Bussineska i yeye gidravlicheskoye prilozheniye [The Boussinesq limit problem and its hydraulic application], In the collection of scientific papers “The Scientific Review of Physics, Mathematics and Engineering Sciences for the 21th Century”, “Prospero” publ. house, Moscow (2015) 3–7.

[4] **M.R. Petrichenko,** Predelnyye zadachi Krokko v teorii struy vyazkoy zhidkosti [The Crocco limit problems in the viscous-fluid jet theory], In the collection of scientific papers “The Scientific Review of Physics, Mathematics and Engineering Sciences for the 21th Century”, “Prospero” publ. house, Moscow, (2016) 10–14.

[5] **D. Nemova, E. Reich, S. Subbotina, et al.,** Heat and mass transfer in a vertical channel under heat-gravitational convection conditions, Experimental Fluid Mechanics Proceedings of the International Conference, Prague (2015) 604–616.

[6] **M. Akdi, M.B. Sedra,** Numerical solution of the Blasius problem, The African Review of Physics 9 (0022) (2014) 165–168.

[7] **F.M. Allan, M.I. Syam,** On the analytic solutions of the nonhomogeneous Blasius problem, Journal of Computational and Applied Mathematics. 182 (2) (2005) 362–371.

[8] **H. Aminikhah,** Analytical approximation to the solution of nonlinear Blasius viscous flow equation by LTNHPM, International Scholarly Research Network, ISRN Mathematical Analysis. 2012 (2012) ID 957473 (10 p).

[9] **H. Aminikhah, S. Kazemi,** Numerical solution of the Blasius viscous flow problem by quartic B-spline method, Hindawi Publishing Corporation, International Journal of Engineering Mathematics. 4 (2016) ID 9014354 (6 p), http://dx.doi.org/10.1155/2016/9014354..

[10] **G. Andrzejczak, M. Nockowska-Rosiak, B. Przeradzk,** A note on Blasius type boundary value problems, Opuscula Math. 33 (1) (2013) 5–17.

[11] **R.C. Bataller,** Numerical comparisons of Blasius and Sakiadis flows, Matematika. 26 (2) (2010) 187–196.

[12] **R. Cortell,** Numerical solutions of the classical Blasius flat-plate problem, Applied Mathematics and Computation. 170 (1) (2005) 706–710.

[13] **C.-W. Chang, C.-S. Liu,** The Lie-group shooting method for boundary layer equations in fluid mechanics, Proceedings of the Conference of Global Chinese Scholars on Hydrodynamics, Beijing, 2006 –07, 103–108.

[14] **T. Fang, C.F. Lee,** A moving-wall boundary layer flow of as slightly rarefied gas free stream over a moving flat plate, Applied Mathematics Letters. 18 (5) (2005) 487–495.

[15] **Tiegang Fang, Wei Liang, Chia-fon F. Lee,** A new solution branch for the Blasius – A shrinking equation sheet problem, Computers and Mathematics with Applications. 56 (12) (2008) 3088–3095.

[16] **R. Fazio,** Blasius problem and Falkner – Skan model: Tцpfer’s algorithm and its extension, Computers & Fluids. 73 (15 March) (2013) 202–209.

[17] **J. Goh, A.A. Majid, A.I.M. Ismail,** A quatic B-spline for second-order singular boundary value problems, Computers and Mathematics with Applications. 64 (12) (2012) 115–120.

[18] **J.-H. He,** A simple perturbation approach to Blasius equation, Applied Mathematics and Computation. 140 (2003) 217–222.

[19] **Chein-Shan Liu, Jiang-Ren Chang,** The Lie-groups shooting method for multiple-solutions of Falkner – Skan equation under suction-injection conditions, International Journal of Non-Linear Mechanics. 43 (2008) 844–851.

[20] **C.-S. Liu,** Cone of non-linear dynamical system and group preserving schemes, International Journal of Non-Linear Mechanics. 36 (7) (2001) 1047–1068.

[21] **C.-S. Liu,** The Lie-group shooting method for non-linear two-point boundary value problems exhibiting multiple solutions, Comput. Model. Eng.

Sci. 13 (2) (2006) 149–163.
[22] **Ding Xu, Xin Guo.** Fixed point analytical method for nonlinear differential equations, Journal of Computational and Nonlinear Dynamics. 8 (1) (2013) 011005 (9 p).
[23] **V.P. Varin,** Flat expansions and their applications, Moscow, KIAM Preprint (23) (2014). URL: http://keldysh.ru/papers/2014/prep2014_23.eng.pdf.
[24] **V.P. Varin,** Flat expansions and their applications, Computational Mathematics and Mathematical Physics. 55 (5) (2015) 797–810.
[25] **V.P. Varin,** Ploskiy asimptoticheskiy ryad Blaziusa [The Blasius's flat asymptotic expansion], In: Mathematics Forum Ser. "Sciences summary, The South of Russia" (2015) 34–47.
[26] **V.P. Varin,** Special solutions to Chazy equation, Moscow, KIAM Preprint (43) (2015). http://keldysh.ru/papers/2015/prep2015_43_eng.pdf.
[27] **V.P. Varin,** Special solutions to Chazy equation, Computational Mathematics and Mathematical Physics. 57 (2) (2017) 211–235.
[28] **V.P. Varin,** Integrirovaniye ODU na Rimanovykh poverkhnostyakh kak vychislitelnyy instrument, [ODE integration on the Riemann surfaces as numerical instrument], In: Teoreticheskiye osnovy konstruirovaniya chislennykh algoritmov i resheniye zadach matematicheskoy fiziki [Theoretical foundation of synthesizing the numerical algorithms and the solution of math-physical problems], Tezisy dokladov XXI Vserossiyskoy konferentsii i Molodezhnoy shkoly-konferentsii, posvyashchennoy pamyati K.I. Babenko [Sci. abstracts of the 21th All-Russian Conf. and the Youth School-Conf. Dedicated to the Memory of K.I. Babenko], KIAM (2016) 73–74.
[29] **Faiz Ahmad,** Application of Crocco – Wang equation to the Blasius problem, Electronic Journal "Technical Acoustics", http://www.ejta.org, 2007, 2.

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