THE GEOMETRICAL EFFECT OF AN ACTIVE ELEMENT CROSS-SECTION ON THE LASER GAIN

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An improved method for calculating the dependence of a laser emission gain on the tube cross-section's geometry has been developed. In this connection the general solution of the Helmholtz equation was considered. But the solution in the form of an infinite series holds the potential for errors. In practice, a researcher has to replace the infinite series by a finite one. Some measures for solving the problems arising in this case were proposed. We have obtained an approximate solution of the Helmholtz equation that is convenient for practice, and a modified method for finding the coefficients of expansion has been developed. The method was tested for some cross-sectional forms that allowed independent theoretical calculation. As a result, the calculations accuracy was demonstrated to improve.

Key words: laser; active element; laser radiation gain; geometry of a tube cross-section

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Introduction

One of the tasks of laser studies is to provide maximum radiation power with a fixed length of the gain medium (GM). The latter usually has a cylindrical shape in gas-discharge lasers (GDL), due to the technology of glass production. The search for reserves of power for GDL has generated many different problems from the early years of the laser era. The question whether the cylindrical geometry of the GM was optimal from the standpoint of energy was one of them. The properties of a rectangular cross-section [1], then of an elliptic cross-section [2] were studied. Experiments with a helium-neon laser with a rectangular cross-section demonstrated an agreement between the model and the laser's actual properties [3]. The encouraging experimental results led to attempts to obtain a generalized theoretical model [4, 5]. Searches based on these generalizations continue in the present day [6, 7].

Notably, the Russian industry did not wait for reliable physical models to be created, starting to produce helium-neon lasers OKG-11 and OKG-12 not only with cylindrical but also with rectangular and elliptical GMs in the 1960s. The expectation was that radiation power would be increased due to higher types of oscillations in the resonator.

The search was not limited to choosing the optimal cross-section of the GM. The models of the optimal longitudinal cross-section were more successful [8, 9]. They were confirmed experimentally and yielded a noticeable increase in the lasing power [10]. However, more factors turned out to affect the power than previously believed. It is also clear from our present-day point of view that the models were not analytical, and the calculations were of an approximate nature. The modern level of computer technology should allow to develop new approaches and discover unexplored opportunities.

This article starts a new series of studies that should (as we hope) allow to gain deeper insight into the nature of the GDL and find additional power reserves.

The geometric part of the laser gain has the following form [1] (in this article we confine ourselves to a constant cross-section along the length of the GM):

$$k = \frac{1}{S_0} \int_V k_0 \cdot f \, dV; \tag{1}$$

the function $f(\mathbf{r})$ describing the spatial distribution of the gain coefficient satisfies the Helmholtz equation in the region V:

$$\Delta f + \lambda^2 f = 0 \tag{2}$$

with the boundary condition

$$f\big|_{\Gamma} = 0. \quad (3)$$

Here Γ is the boundary of the region in which the solution is sought; S_0 is the cross-sectional area of the tube; k_0 is the gain on the axis of the system.

General solution of the Helmholtz equation

Let us examine Eq. (2) in cylindrical coordinates (r, ϕ, z), assuming from symmetry considerations that there is no dependence on the z coordinate. Using the variable separation method, we find that the general solution of the equation for polar angle periodicity and a solution bounded in the neighborhood of $r = 0$ is a set of functions

$$f(r,\phi) = a_{k1} J_k(\lambda r) \cdot \cos(k\phi) + \\ + a_{k2} J_k(\lambda r) \cdot \sin(k\phi),$$

where $J_k(\lambda r)$ are the Bessel functions of order k; k = 0, 1, 2, ... (there is a single function $a_0 J_0(\lambda r)$ for $k = 0$).

In this case, in general, boundary condition (3) with an arbitrary form of the boundary Γ of the region V (that is, with an arbitrary cross-section of the tube) is satisfied by a series of similar functions, rather than by any individual function for some fixed value of k, i.e., in the presence of condition (3), the general solution of Eq. (2) has the form

$$f(r,\phi) = \sum_{k=0}^{\infty} \{a_{k1} J_k(\lambda r) \cdot \cos(k\phi) + \\ + a_{k2} J_k(\lambda r) \cdot \sin(k\phi)\}. \quad (4)$$

A similar method for solving the boundary value problem in the form of expansions of the solution in exact solutions of the corresponding differential equation is often referred to as the Trefftz method in technical literature [11].

In this paper, we propose our own modification of this method, which allows to obtain solutions with a high accuracy and relatively low computational complexity.

Approximate solution of the Helmholtz equation

In practical calculations, we have to replace the infinite series with a finite one, taking a certain number of terms in expression (4) and obtaining the function $f^n(r,\phi)$:

$$f^n(r,\phi) = \sum_{k=0}^{n} \{a_{k1} J_k(\lambda r) \cdot \cos(k\phi) + \\ + a_{k2} J_k(\lambda r) \cdot \sin(k\phi)\}. \quad (5)$$

Function (5), the same as function (4), satisfies Eq. (2) exactly (since each term satisfies this equation), but only approximately satisfies boundary condition (3).

Let us denote the corresponding functions from expression (5) as ξ_k, i.e.,

$$\zeta_k(\lambda r, \phi) = J_k(\lambda r) \cdot \cos(k\phi)$$

or

$$\zeta_k(\lambda r, \phi) = J_k(\lambda r) \cdot \sin(k\phi),$$

and normalize our function f (and, respectively, the function $f^n(r,\phi)$) by unity for r = 0 (Eq. (2) and boundary condition (3) are homogeneous).

Since $J_0(0) = 1$, $J_k(0) = 0$ with $k > 0$, then $a_0 = 1$ and

$$f^n(r,\phi) = \zeta_0(\lambda r, \phi) + \sum_{k=1}^{n} a_k \zeta_k(\lambda r, \phi). \quad (6)$$

Let us now formulate what we mean by stating that boundary condition (3) is satisfied approximately.

We choose some N points on the boundary Γ

$$\xi_1, \xi_2, \ldots, \xi_N$$

and impose the condition that the sum of the values of the function f^n be equal to zero at these points:

$$\sum_{j=1}^{N} f^n(\xi_j) = 0, \ \xi_j \in \Gamma.$$

In practical calculations, we can also generally satisfy the latter equality only approximately, so we should rigorously formulate the condition in the following manner: the values of the parameter λ and the a_k coefficients in equality (6) should be chosen such that the absolute value of the sum is less than some given, very small value Δ:

$$\left| \sum_{j=1}^{N} f^n(\xi_j) \right| < \Delta, \ \xi_j \in \Gamma. \quad (7)$$

It is extremely ineffective to go through all possible values of the parameter λ and the a_k coefficients, so we will search for them by the approximation method.

The first approximation for the parameter λ. In the first approximation, we impose the following condition on λ:

$$\sum_{j=1}^{N} \zeta_0(\xi_j) \equiv \sum_{j=1}^{N} J_0(\lambda r_j) = 0, \qquad (8)$$

where r_j is the corresponding value of the polar coordinate r of the point ξ_j.

In practice, we can either demand that the sum be equal to zero in condition (8) approximately with some accuracy, or replace the Bessel function J_0 with an approximate one, for which equation (8) can be solved exactly. For example, we can interpolate it with a second-degree polynomial (the second degree is convenient because in this case the equation is square and easily solvable; solution would be problematic for higher degrees). Interpolating the function with a polynomial of only the second degree is completely justified, since this is only the first approximation for the parameter λ, which will be refined further, and inappropriate choice of the first approximation can lead only to computational complexity of the algorithm, but will not worsen the final accuracy. So, if $[r_{\min}, r_{\max}]$ is the variation range of the coordinate r for the boundary Γ, we can construct, for example, the Lagrange interpolating polynomial of the second degree for the function $\zeta_0(r)$ in the nodes r_0, r_1, r_2 from the range $[r_{\min}, r_{\max}]$ [12]:

$$P(r) = \zeta_0(r_0)\frac{(r-r_1)(r-r_2)}{(r_0-r_1)(r_0-r_2)} + \zeta_0(r_1)\frac{(r-r_0)(r-r_2)}{(r_1-r_0)(r_1-r_2)} + \zeta_0(r_2)\frac{(r-r_0)(r-r_1)}{(r_2-r_0)(r_2-r_1)} = ar^2 + br + c,$$

where

$$a = \frac{\zeta_0(r_0)}{(r_0-r_1)(r_0-r_2)} + \frac{\zeta_0(r_1)}{(r_1-r_0)(r_1-r_2)} + \frac{\zeta_0(r_2)}{(r_2-r_0)(r_2-r_1)};$$

$$b = -\zeta_0(r_0)\frac{(r_1+r_2)}{(r_0-r_1)(r_0-r_2)} - \zeta_0(r_1)\frac{(r_0+r_2)}{(r_1-r_0)(r_1-r_2)} - \zeta_0(r_2)\frac{(r_0+r_1)}{(r_2-r_0)(r_2-r_1)};$$

$$c = \frac{\zeta_0(r_0)\cdot r_1\cdot r_2}{(r_0-r_1)(r_0-r_2)} + \frac{\zeta_0(r_1)\cdot r_0\cdot r_2}{(r_1-r_0)(r_1-r_2)} + \frac{\zeta_0(r_2)\cdot r_0\cdot r_1}{(r_2-r_0)(r_2-r_1)}.$$

We can take the roots of the corresponding Chebyshev polynomials as interpolation nodes:

$$r_k = \frac{r_{\max}+r_{\min}}{2} + \frac{r_{\max}-r_{\min}}{2}\cos\left(\frac{2k+1}{2n+2}\pi\right),$$

where $k = 0, 1, 2, ..., n$ and $n = 2$ in our case.

Substituting this into (9), we obtain an equation with respect to the parameter λ:

$$\lambda^2 \cdot a\sum_{j=1}^{N} r_j^2 + \lambda \cdot b\sum_{j=1}^{N} r_j + \sum_{j=1}^{N} c = 0, \qquad (9)$$

from which we find the first approximation for λ as the value of the corresponding positive root.

If it turns out that Eq. (9) has no positive roots for a given choice of points $\xi_1, \xi_2, \ldots, \xi_N$ on the boundary Γ (that is, the corresponding values of r_j), then, since the choice of the corresponding boundary points in the problem is not rigid, the above set of points can be changed and the algorithm can be repeated again. If, on the other hand, either the set of boundary points is rigidly fixed for some reason or we still cannot obtain the parameter λ after several changes of boundary points, then we can take, as a first approximation for λ, such a value that

$$J_0(\lambda r_{mean}) = 0,$$

where r_{mean} is the mean value of r in the cross-section of the tube, i.e., $\lambda = \lambda_1 / r_{mean}$, where $\lambda_1 = 2.4048$ is the first root of the function $J_0(x)$.

In this case, we simply have to increase the interval where we search for the next approximations of the parameter λ in subsequent calculations.

First approximation of the coefficients a_k. It follows from boundary condition (3) that series (6)

$$\zeta_0(\lambda r,\phi)+\sum_{k=1}^{n} a_k\zeta_k(\lambda r,\phi)$$

is equal to zero on the boundary Γ, that is, the following equality should hold true on the boundary Γ:

$$-\zeta_0(\lambda r,\phi)=\sum_{k=1}^{n} a_k\zeta_k(\lambda r,\phi);\ (r,\phi)\in\Gamma. \quad (10)$$

Formula (10) can be interpreted as an approximation of the function $-\zeta_0$ on the boundary Γ by a linear combination of linearly independent functions ζ_k. However, it is more convenient to perform such an approximation through orthonormal (on the boundary Γ) functions, so let us move on from the set of linearly independent functions $\{\zeta_1,\zeta_2,\ \ldots,\zeta_n\}$ to a set of functions $\{\Psi_1,\Psi_2,\ \ldots,\Psi_n\}$ that are orthonormal on the boundary Γ (in fact, we only determine these functions on the boundary).

A transition to an orthonormal set can be made, for example, through the Gram – Schmidt process. In this case, the scalar product of the functions on the boundary Γ is naturally determined as

$$\langle\zeta_k,\zeta_s\rangle=\sum_{j=1}^{N}\zeta_k(\lambda r_j,\phi_j)\cdot\zeta_s(\lambda r_j,\phi_j);$$

$$\langle\Psi_k,\Psi_s\rangle=\sum_{j=1}^{N}\Psi_k(r_j,\phi_j)\cdot\Psi_s(r_j,\phi_j);$$

$$\langle\zeta_k,\Psi_s\rangle=\sum_{j=1}^{N}\zeta_k(\lambda r_j,\phi_j)\cdot\Psi_s(r_j,\phi_j),$$

and the norm of an arbitrary function Y on the boundary Γ is determined as follows:

$$\|Y\|=\sqrt{\langle Y,Y\rangle}.$$

The Gram – Schmidt process involves constructing non-normalized orthogonal functions $\{\tilde{Y}_k\}$ at first and then normalizing them:

$$\Psi_k(r,\varphi)=\tilde{Y}_k(r,\ \varphi)/\|\tilde{Y}_k\|.$$

The construction is carried out as follows. The first function ζ_1 is chosen as the first function $\tilde{Y}_1$. More precisely, the values of this function $\tilde{Y}_1$ on the set of points $\xi_1,\ \xi_2,\ \ldots,\ \xi_N$ on the boundary Γ are determined through the values of the first function ζ_1 on the same set of points:

$$\tilde{Y}_1(\xi_j)\equiv\tilde{Y}_1(r_j,\phi_j)=\zeta_1(\lambda r_j,\phi_j),\ \ j=1,2,\ldots,N;$$

$$\Psi_1(\xi_j)=\tilde{Y}_1(r_j,\phi_j)/\|\tilde{Y}_1\|.$$

The values of the next functions are found successively by the formulae:

$$\tilde{Y}_i(\xi_j)\equiv\tilde{Y}_i(r_j,\phi_j)=$$

$$=\zeta_i(\lambda r_j,\phi_j)-\sum_{k=1}^{i-1}\frac{\langle\zeta_i,\tilde{Y}_k\rangle}{\langle\tilde{Y}_k,\tilde{Y}_k\rangle}\tilde{Y}_k(r_j,\phi_j), \quad (11)$$

$$j=1,2,\ \ldots,N,\ i=2,\ \ldots,\ n;$$

$$\Psi_i(\xi_j)=\tilde{Y}_i(r_j,\phi_j)/\|\tilde{Y}_i\|.$$

The resulting set of functions Ψ_i is orthonormal on the boundary Γ, and therefore any function, including the function $-\zeta_0$, can be expanded in a series (approximated by a linear combination) of these functions:

$$-\zeta_0(\lambda r_j,\phi_j)=\sum_{k=1}^{n}\beta_k\Psi_k(r_j,\phi_j); \quad (12)$$
$$j=1,2,\ \ldots,N,(r_j,\phi_j)\in\Gamma.$$

Since the system $\{\Psi_1,\Psi_2,\ \ldots,\Psi_n\}$ is orthonormal, it is easy to find the expansion coefficients β_k:

$$\beta_k=-\langle\zeta_0,\Psi_k\rangle=-\sum_{j=1}^{N}\zeta_0(\lambda r_j,\phi_j)\cdot\Psi_k(r_j,\phi_j).$$

Comparing equations (10) and (12), we obtain the equality

$$\beta_1\Psi_1(r_j,\phi_j)+\beta_2\Psi_2(r_j,\phi_j)+\ldots+\beta_n\Psi_n(r_j,\phi_j)=$$
$$=a_1\zeta_1(\lambda r_j,\phi_j)+a_2\zeta_2(\lambda r_j,\phi_j)+\ldots \quad (13)$$
$$\ldots+a_n\zeta_n(\lambda r_j,\phi_j),$$

which must be satisfied for all points j = 1, 2, ..., N.

On the other hand, by virtue of their construction, each of the functions Ψ_k (k = 1, 2,..., n) is a linear combination of functions $\{\zeta_1,\zeta_2,\ \ldots,\zeta_n\}$. Now, equating the coefficients for the same functions $\zeta_k(\lambda r,\phi)$ on the left-hand and on the right-hand sides in equality (13), we obtain expressions for the coefficients a_k. Substituting the first approximation for λ in these expressions, we obtain the first approximation for the coefficients a_k.

These expressions for the coefficients a_k are rather cumbersome and become more complicated with increasing number n in formula (6). The expressions for a_k for the case $n = 5$ are given in Appendix 1.

Modified method for finding the coefficients a_k

The resulting set of functions $\{\Psi_1, \Psi_2, \ldots, \Psi_n\}$ may not be exactly orthogonal in numerical calculations, due to rounding errors, since the Gram – Schmidt process is numerically unstable [13]. However, this process can be made more computationally stable by modifying it. Let us describe the proposed modification procedure.

The first two functions Ψ_1, Ψ_2 are found in a similar way:

$$\tilde{Y}_1(\xi_j) = \zeta_1(\lambda r_j, \phi_j),\ j = 1, 2, \ldots, N;$$

$$\Psi_1(\xi_j) = \tilde{Y}_1(r_j, \phi_j) / \| \tilde{Y}_1 \|;$$

$$\tilde{Y}_2(\xi_j) = \zeta_2(\lambda r_j, \phi_j) - \frac{\langle \zeta_2, \tilde{Y}_1 \rangle}{\langle \tilde{Y}_1, \tilde{Y}_1 \rangle} \tilde{Y}_1(r_j, \phi_j),$$

$$j = 1, 2, \ldots, N;$$

$$\Psi_2(\xi_j) = \tilde{Y}_2(r_j, \phi_j) / \| \tilde{Y}_2 \|.$$

After that, the functions are found as follows:

$$\zeta_3^{(1)}(\lambda r_j, \phi_j) = \zeta_3(\lambda r_j, \phi_j) - \frac{\langle \zeta_3, \tilde{Y}_1 \rangle}{\langle \tilde{Y}_1, \tilde{Y}_1 \rangle} \tilde{Y}_1(r_j, \phi_j),$$

$$j = 1, 2, \ldots, N;$$

$$\tilde{Y}_3(r_j, \phi_j) = \zeta_3^{(1)}(\lambda r_j, \phi_j) -$$

$$- \frac{\langle \zeta_3^{(1)}, \tilde{Y}_2 \rangle}{\langle \tilde{Y}_2, \tilde{Y}_2 \rangle} \tilde{Y}_2(r_j, \phi_j),\ j = 1, 2, \ldots, N;$$

$$\Psi_3(\xi_j) = \tilde{Y}_3(r_j, \phi_j) / \| \tilde{Y}_3 \|$$

and in the general case $\tilde{Y}_k$ are found by the following algorithm:

$$\zeta_k^{(1)}(\lambda r_j, \phi_j) = \zeta_k(\lambda r_j, \phi_j) -$$

$$- \frac{\langle \zeta_k, \tilde{Y}_1 \rangle}{\langle \tilde{Y}_1, \tilde{Y}_1 \rangle} \tilde{Y}_1(r_j, \phi_j),\ j = 1, 2, \ldots, N;$$

$$\zeta_k^{(2)}(\lambda r_j, \phi_j) = \zeta_k^{(1)}(\lambda r_j, \phi_j) -$$

$$- \frac{\langle \zeta_k^{(1)}, \tilde{Y}_2 \rangle}{\langle \tilde{Y}_2, \tilde{Y}_2 \rangle} \tilde{Y}_2(r_j, \phi_j),\ \ j = 1, 2, \ldots, N;$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots;$$

$$\zeta_k^{(k-2)}(\lambda r_j, \phi_j) = \zeta_k^{(k-3)}(\lambda r_j, \phi_j) -$$

$$- \frac{\langle \zeta_k^{(k-3)}, \tilde{Y}_{k-2} \rangle}{\langle \tilde{Y}_{k-2}, \tilde{Y}_{k-2} \rangle} \tilde{Y}_{k-2}(r_j, \phi_j),\ \ j = 1, 2, \ldots, N;$$

$$\tilde{Y}_k(r_j, \phi_j) = \zeta_k^{(k-2)}(\lambda r_j, \phi_j) - \qquad (14)$$

$$- \frac{\langle \zeta_k^{(k-2)}, \tilde{Y}_{k-1} \rangle}{\langle \tilde{Y}_{k-1}, \tilde{Y}_{k-1} \rangle} \tilde{Y}_{k-1}(r_j, \phi_j),\ \ j = 1, 2, \ldots, N;$$

$$\Psi_k(\xi_j) = \tilde{Y}_k(r_j, \phi_j) / \| \tilde{Y}_k \|.$$

This modified algorithm is equivalent to the one described above provided that the calculations are absolutely exact (i.e., functions (14) coincide with the corresponding functions (11) for absolutely exact calculations). The coefficients a_k are found from equality (13) in a similar way. The expressions for a_k for the case $n = 6$ are given in Appendix 2.

In order to decide whether to use the standard or the modified orthogonalization method, we should verify whether the equalities

$$\langle \Psi_i, \Psi_k \rangle = \delta_{ik},$$

where δ_{ik} is the Kronecker symbol, hold true.

Our calculations for the ellipse with $N = 400$ show that these equalities are satisfied more accurately by approximately 14 orders of magnitude for the modified method (using 64-bit floating-point numbers) even for values of i and k equal to 5 and 6.

Subsequent approximations and calculation of the gain coefficient

After we have found the parameter λ and the coefficients a_k in the first approximations, we can compose the expression for the function $f^n(r, \phi)$ by formula (6) and find sum (7) in the first approximation:

$$\Delta_1 = \left| \sum_{j=1}^{N} f^n(\xi_j) \right|,\ \xi_j \in \Gamma.$$

Next, we vary the value of the parameter λ, changing it by $\delta\lambda$. We construct a new system of orthonormal functions for this new value $\lambda + \delta\lambda$ by formulae (11) or (14); for this new system we find the new expansion coefficients β_k and then find the new values of the coefficients a_k from equality (13). We find sum (7) from the new values of a_k; this is Δ_2. Then we repeat the whole process. Thus, by searching through the possible values of λ in the neighborhood of the first approximation, we finally choose the value with which sum (7) reaches the minimum. The number of points N on the boundary can also be changed (keeping in mind that the first approximation has to be found again in this case) to achieve the required accuracy.

We should note the following. According to theory, the solution of Eq. (2) with boundary condition (3) exists not for some single value, but for a set of values of λ (the so-called set of eigenvalues of the problem). The first eigenvalue of λ makes the main contribution to the gain. If we greatly increase the search interval for the first approximation, then in the subsequent approximations we risk obtaining not the first but the subsequent eigenvalues, since the approximate numerical value of sum (7) can be closer to zero for the subsequent eigenvalues than for the first one. In view of this, the dependence of sum (7) on the value of λ should be controlled to verify that the first eigenvalue has been obtained.

After we have determined the parameter λ and the coefficients a_k with the required accuracy, we can determine the gain coefficient (here and below we choose the tube length equal to unity):

$$k = \frac{1}{S_0}\iint_S k_0 f^n(r,\phi) r dr d\phi = \frac{1}{S_0}\Big(\iint_S k_0 \zeta_0(\lambda r,\phi) r dr d\phi + \sum_{k=1}^{n} a_k \iint_S k_0 \zeta_k(\lambda r,\phi) r dr d\phi\Big), \quad (15)$$

where S is the cross-section of the tube with the area S_0.

Let us demonstrate the application of this method for cross-sections of different shapes.

The case of a rectangular cross-section of the tube

Let the sides of the rectangle be equal to a and b, with $b \le a$. We select the origin of the polar coordinate system at the center of the rectangle and direct the polar axis (from which the polar angle ϕ is measured) parallel to the larger side. Let θ be half the smaller angle between the diagonals, and then the larger angle between the diagonals is equal to $\pi - 2\theta$. Obviously $\mathrm{tg}\theta = b/a$. Then with the coordinate system we have chosen the coordinates (r,ϕ) of the vertices of the rectangle are expressed as

$$\left(\frac{\sqrt{a^2+b^2}}{2}, \mathrm{arctg}\frac{b}{a}\right),$$

$$\left(\frac{\sqrt{a^2+b^2}}{2}, \pi - \mathrm{arctg}\frac{b}{a}\right),$$

$$\left(\frac{\sqrt{a^2+b^2}}{2}, \pi + \mathrm{arctg}\frac{b}{a}\right),$$

$$\left(\frac{\sqrt{a^2+b^2}}{2}, -\mathrm{arctg}\frac{b}{a}\right).$$

Since in polar coordinates the equation of a straight line has the form

$$r = p / \cos(\phi - \alpha),$$

where p is the length of the perpendicular to the straight line from the origin, α is the polar angle of this perpendicular, then it follows from formula (15) that the gain coefficient has the form

$$k = \frac{1}{ab}\left\{ \int_{-\mathrm{arctg}\frac{b}{a}}^{\mathrm{arctg}\frac{b}{a}} d\phi \int_0^{\frac{a}{2\cos\phi}} r dr k_0 f^n(r,\phi) + \int_{\mathrm{arctg}\frac{b}{a}}^{\pi-\mathrm{arctg}\frac{b}{a}} d\phi \int_0^{\frac{b}{2\sin\phi}} r dr k_0 f^n(r,\phi) + \int_{\pi-\mathrm{arctg}\frac{b}{a}}^{\pi+\mathrm{arctg}\frac{b}{a}} d\phi \int_0^{\frac{-a}{2\cos\phi}} r dr k_0 f^n(r,\phi) + \int_{\pi+\mathrm{arctg}\frac{b}{a}}^{2\pi-\mathrm{arctg}\frac{b}{a}} d\phi \int_0^{\frac{-b}{2\sin\phi}} r dr k_0 f^n(r,\phi) \right\}.$$

With the coordinate system we have chosen, obviously, there is a symmetry when replacing ϕ with $-\phi$, which means that the function $f^n(r,\phi)$ must be even with respect to ϕ. Similarly, nothing should change in symmetry when replacing ϕ with $\pi-\phi$. This means that the expression

$$\zeta_k(\lambda r,\phi) = J_{2k}(\lambda r)\cdot\cos(2k\phi)$$

should be taken as the function $\zeta_k(\lambda r,\phi)$, i.e., formula (6) takes the following form:

$$f^n(r,\phi) = J_0(\lambda r) + \sum_{k=1}^{n} a_{2k} J_{2k}(\lambda r)\cdot\cos(2k\phi). \quad (16)$$

The results of the calculations based on the obtained formulae for the rectangular cross-section of the tube for different side ratios *a*/*b* (*b* = 1) for *N* = 400 points on the boundary are given in Table 1. For a square, with *a*/*b* = 1, there is additional symmetry when replacing ϕ with $\phi+\pi/2$, so only the coefficients a_{4k} differ from zero, which is well-confirmed by the calculation.

However, Eq. (2) with boundary condition (3) for a rectangle can be solved exactly in the Cartesian coordinate system. Writing boundary condition (3) in the form

$$f(a/2,y) = f(-a/2,y) = f(x,b/2) = \\ = f(x,-b/2) = 0$$

and using the variable separation method again, we obtain that for the spatial distribution of the gain *f*(*x*,*y*) in the case of a tube with a rectangular cross-section, the general solution of Eq. (2) satisfying condition (3) is the function

$$f(x,y) = C\cos\left(\frac{n\pi}{a}x\right)\cdot\cos\left(\frac{m\pi}{b}y\right),$$

and obtain the following expression for the value of the parameter λ:

$$\lambda^2_{n,m} = \frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2},\ n = 1,2,3...;\ m = 1,2,3... .$$

Normalizing the function *f* to unity at the center of the cross-section, we obtain *C* = 1. For comparison with the results of the above calculations (data for the first eigenvalue of λ), it is convenient to take the component for which *n* = *m* = 1. Then the gain coefficient for a tube with a rectangular cross-section can be calculated by formula (1):

$$k = \frac{1}{ab}\int_{-\frac{a}{2}}^{\frac{a}{2}}\int_{-\frac{b}{2}}^{\frac{b}{2}} k_0\cos\frac{\pi x}{a}dx\cdot\cos\frac{\pi y}{b}dy = \\ = \frac{4}{\pi^2}k_0 = 0.405k_0.$$

Comparing the obtained value of the coefficient with that in Table 1 (bottom column) shows excellent agreement between the results of applying our method (finding both the gain coefficient and the parameter λ) and the exact solution.

The case of a circular cross-section of the tube

An exact solution can also be obtained for this cross-section. If the center of the circle is taken as the origin of the polar coordinate system, the sought-for function does not depend

Table 1

Calculated values of the main parameters of the system depending on the side ratio of the rectangular cross-section of the laser tube

a/*b*	λ	a_2	a_4	a_6	a_8	a_{10}	a_{12}
1.00	4.44	0.00	−2.00	0.00	2.00	0.00	−1.93
1.10	4.25	0.19	−1.96	−0.56	1.86	0.92	−1.69
1.25	4.02	0.44	−1.81	−1.23	1.27	1.79	−0.51
1.50	3.78	0.77	−1.41	−1.85	−0.02	1.84	1.41
1.75	3.62	1.02	−0.97	−2.00	−1.06	0.92	2.04
2.00	3.51	1.20	−0.56	−1.87	−1.69	−0.16	1.60
The normalized calculated value of the gain coefficient k/k_0 = 0.405 (k_0 is the gain on the axis of the system) for all given values of the ratio *a*/*b*							

on the polar angle ϕ, and after replacing the argument $x = \lambda r$, Eq. (2) becomes an equation for the Bessel function of zero order. Its general solution has the form

$$f(r,\phi) = f(r) = a_0 J_0(\lambda r).$$

Let us normalize the function f to unity again at the center of the cross-section, and then we obtain that $f(r) = J_0(\lambda r)$.

Boundary condition (3) yields the value $J_0(\lambda a) = 0$ for a circle of radius a, and it follows then that the product λa takes the values

$$\lambda a = \lambda_1, \lambda_2, \lambda_3, \ldots ,$$

where λ_k is the kth root of the function $J_0(x)$.

Since the main contribution to the spatial distribution of the gain is made by the first eigenfunction, $\lambda a = \lambda_1 = 2.4048$. Then by formula (1) we obtain the following expression for the gain coefficient:

$$k = \frac{1}{\pi a^2} \int_0^{2\pi} d\phi \int_0^a r dr k_0 J_0\left(\frac{\lambda_1}{a} r\right) =$$

$$= \frac{2k_0}{\lambda_1^2} J_1\left(\frac{\lambda_1}{a} r\right) \cdot \frac{\lambda_1}{a} r \Bigg|_0^a = \frac{2k_0}{\lambda_1} J_1(\lambda_1) = 0.432 k_0.$$

To test our algorithm, we performed calculations for a circular cross-section of the tube according to the general formula (15), which in this case has the form

$$k = \frac{1}{\pi a^2} \int_0^{2\pi} d\phi \int_0^a r dr k_0 f^n(r,\phi).$$

Since symmetry can be used again to replace the angle ϕ with the angle $-\phi$ and the angle $\pi + \phi$, we can once again take

$$\zeta_k(\lambda r, \phi) = J_{2k}(\lambda r) \cdot \cos(2k\phi),$$

as the function $\zeta_k(\lambda r, \phi)$, and use formula (16) for the function $f^n(r,\phi)$.

We should note that the circular cross-section is that very rare case when, obviously, the first approximation of λ cannot be found from formula (9); therefore, we choose the parameter $\lambda = \lambda_1 / a$ as the first approximation (based on the reasoning given above in the section “First approximation of λ ”).

The results of the calculation for $N = 400$ points on the boundary are provided by the equalities

$$a_2 = a_4 = a_6 = a_8 = a_{10} = a_{12} = 0$$

being fulfilled with a high accuracy (the coefficients a_k are of the order of $10^{-16} - 10^{-21}$); the values of the parameter λ and the gain coefficient k also agree well with the theoretical values.

The case of the elliptical cross-section of the tube

An analytically exact expression for the laser gain coefficient cannot be obtained in this case. Let us use the method we propose.

We take the center of the ellipse as the origin of the polar coordinate system, and direct the polar axis along the ellipse’s semi-major axis. Let c and d be the semi-axes of the ellipse ($d \le c$), then the ellipse equation has the form

Table 2

Calculated values of the main parameters of the system as a function of the semi-axes ratio of the elliptical cross-section of the laser tube

c/d	k/k_0	λ	a_2	a_4	a_6	a_8	a_{10}
1.00	0.432	2.41	$<10^{-6}$	$<10^{-6}$	$<10^{-6}$	$<10^{-6}$	$<10^{-6}$
1.10	0.431	2.30	0.14	$2\cdot10^{-3}$	$2\cdot10^{-5}$	$<10^{-6}$	$<10^{-6}$
1.25	0.430	2.18	0.32	0.01	$2\cdot10^{-4}$	$<10^{-6}$	$<10^{-6}$
1.50	0.424	2.04	0.56	0.04	$2\cdot10^{-3}$	$3\cdot10^{-5}$	$<10^{-6}$
1.75	0.416	1.95	0.75	0.08	$4\cdot10^{-3}$	$1\cdot10^{-4}$	$<10^{-6}$
2.00	0.408	1.89	0.89	0.13	0.01	$4\cdot10^{-4}$	$1\cdot10^{-5}$

Notes. 1. The value $c/d = 1.00$ is for the circular cross-section of the tube. 2. All values of $a_{12} < 10^{-6}$. 3. The number of points on the boundary is $N = 400$.

$$r = \frac{cd}{\sqrt{c^2 \sin^2 \phi + d^2 \cos^2 \phi}}.$$

Then, according to formula (15), the gain coefficient follows the expression

$$k = \frac{1}{\pi cd} \int_0^{2\pi} d\phi \int_0^{\frac{cd}{\sqrt{c^2 \sin^2 \phi + d^2 \cos^2 \phi}}} r dr k_0 f^n(r, \phi).$$

Symmetry is again preserved for the ellipse when replacing the angle ϕ with the angle $-\phi$ and $\pi + \phi$, and therefore, we should once again take

$$\zeta_k(\lambda r, \phi) = J_{2k}(\lambda r) \cdot \cos(2k\phi)$$

as the function $\zeta_k(\lambda r, \phi)$ and use formula (16) for the function $f^n(r, \phi)$.

The results of the calculations for an elliptical cross-section of a laser tube with different semi-axes ratios c/d (d = 1) for N = 400 points on the boundary are given in Table 2 (also listing the result for the circular cross-section with c/d = 1).

Conclusion

In this study, we have proposed an improved method for calculating the gain coefficient of laser radiation as a function of the geometry of the tube's cross-section. The method allows to obtain more general and accurate results in comparison with the currently existing options. In the future, we plan to use this method to calculate this coefficient for other shapes of the cross-section of the laser gain medium.

Appendix 1

Expressions for the coefficients a_k with n = 5 for the classical Gram – Schmidt method

Let us introduce the following notations:

$$\alpha = \frac{1}{\| \zeta_1 \|}; \ \varepsilon = \frac{\langle \zeta_2, \Psi_1 \rangle}{\| \Psi_1 \|^2}; \ \gamma_1 = \frac{\langle \zeta_3, \Psi_1 \rangle}{\| \Psi_1 \|^2};$$

$$\gamma_2 = \frac{\langle \zeta_3, \tilde{Y}_2 \rangle}{\| \tilde{Y}_2 \|^2}; \delta_1 = \frac{\langle \zeta_4, \Psi_1 \rangle}{\| \Psi_1 \|^2}; \ \delta_2 = \frac{\langle \zeta_4, \tilde{Y}_2 \rangle}{\| \tilde{Y}_2 \|^2};$$

$$\delta_3 = \frac{\langle \zeta_4, \tilde{Y}_3 \rangle}{\| \tilde{Y}_3 \|^2}; \ \omega_1 = \frac{\langle \zeta_5, \Psi_1 \rangle}{\| \Psi_1 \|^2}; \omega_2 = \frac{\langle \zeta_5, \tilde{Y}_2 \rangle}{\| \tilde{Y}_2 \|^2};$$

$$\omega_3 = \frac{\langle \zeta_5, \tilde{Y}_3 \rangle}{\| \tilde{Y}_3 \|^2}; \ \omega_4 = \frac{\langle \zeta_5, \tilde{Y}_4 \rangle}{\| \tilde{Y}_4 \|^2};$$

$$\beta_1 = -\langle \zeta_0, \Psi_1 \rangle; \beta_2 = -\langle \zeta_0, \Psi_2 \rangle;$$

$$\beta_3 = -\langle \zeta_0, \Psi_3 \rangle; \beta_4 = -\langle \zeta_0, \Psi_4 \rangle;$$

$$\beta_5 = -\langle \zeta_0, \Psi_5 \rangle.$$

With the above notations, the coefficients a_k take the following form:

$$a_5 = \frac{\beta_5}{\| \tilde{Y}_5 \|}; a_4 = \frac{\beta_4}{\| \tilde{Y}_4 \|} - \frac{\beta_5 \omega_4}{\| \tilde{Y}_5 \|};$$

$$a_3 = \frac{\beta_3}{\| \tilde{Y}_3 \|} - \frac{\beta_4 \delta_3}{\| \tilde{Y}_4 \|} + \frac{\beta_5 \omega_4 \delta_3 - \beta_5 \omega_3}{\| \tilde{Y}_5 \|};$$

$$a_2 = \frac{\beta_2}{\| \tilde{Y}_2 \|} - \frac{\beta_3 \gamma_2}{\| \tilde{Y}_3 \|} + \frac{-\beta_4 \delta_2 + \beta_4 \delta_3 \gamma_2}{\| \tilde{Y}_4 \|} + (\beta_5 \omega_4 \delta_2 - \beta_5 \omega_4 \delta_3 \gamma_2 + \beta_5 \omega_3 \gamma_2 - \beta_5 \omega_2) / \| \tilde{Y}_5 \|;$$

$$a_1 = \beta_1 \alpha - \frac{\beta_2 \varepsilon \alpha}{\| \tilde{Y}_2 \|} + \frac{-\beta_3 \gamma_1 \alpha + \beta_3 \gamma_2 \varepsilon \alpha}{\| \tilde{Y}_3 \|} + (-\beta_4 \delta_1 \alpha + \beta_4 \delta_2 \varepsilon \alpha + \beta_4 \delta_3 \gamma_1 \alpha - \beta_4 \delta_3 \gamma_2 \varepsilon \alpha) / \| \tilde{Y}_4 \| + (\beta_5 \omega_4 \delta_1 \alpha - \beta_5 \omega_4 \delta_2 \varepsilon \alpha - \beta_5 \omega_4 \delta_3 \gamma_1 \alpha + \beta_5 \omega_4 \delta_3 \gamma_2 \varepsilon \alpha - \beta_5 \omega_3 \gamma_2 \varepsilon \alpha + \beta_5 \omega_3 \gamma_1 \alpha + \beta_5 \omega_2 \varepsilon \alpha - \beta_5 \omega_1 \alpha) / \| \tilde{Y}_5 \|.$$

Appendix 2

Expressions for the coefficients a_k with n = 6 for the modified Gram – Schmidt method

Let us introduce the following notations:

$$\varepsilon = \frac{\langle \zeta_2, \tilde{Y}_1 \rangle}{\| \tilde{Y}_1 \|^2}; \gamma_1 = \frac{\langle \zeta_3, \tilde{Y}_1 \rangle}{\| \tilde{Y}_1 \|^2}; \gamma_2 = \frac{\langle \zeta_3^{(1)}, \tilde{Y}_2 \rangle}{\| \tilde{Y}_2 \|^2};$$

$$\delta_1 = \frac{\langle \zeta_4, \tilde{Y}_1 \rangle}{\| \tilde{Y}_1 \|^2}; \delta_2 = \frac{\langle \zeta_4^{(1)}, \tilde{Y}_2 \rangle}{\| \tilde{Y}_2 \|^2};$$

$$\delta_3 = \frac{\langle \zeta_4^{(2)}, \tilde{Y}_3 \rangle}{\| \tilde{Y}_3 \|^2}; \omega_1 = \frac{\langle \zeta_5, \tilde{Y}_1 \rangle}{\| \tilde{Y}_1 \|^2};$$

$$\omega_2 = \frac{\langle \zeta_5^{(1)}, \tilde{Y}_2 \rangle}{\| \tilde{Y}_2 \|^2}; \omega_3 = \frac{\langle \zeta_5^{(2)}, \tilde{Y}_3 \rangle}{\| \tilde{Y}_3 \|^2};$$

$$\omega_4 = \frac{\langle \zeta_5^{(3)}, \tilde{Y}_4 \rangle}{\| \tilde{Y}_4 \|^2}; \rho_1 = \frac{\langle \zeta_6, \tilde{Y}_1 \rangle}{\| \tilde{Y}_1 \|^2};$$

$$\rho_2 = \frac{\left\langle \zeta_6^{(1)}, \tilde{Y}_2 \right\rangle}{\| \tilde{Y}_2 \|^2}; \rho_3 = \frac{\left\langle \zeta_6^{(2)}, \tilde{Y}_3 \right\rangle}{\| \tilde{Y}_3 \|^2};$$

$$\rho_4 = \frac{\left\langle \zeta_6^{(3)}, \tilde{Y}_4 \right\rangle}{\| \tilde{Y}_4 \|^2}; \rho_5 = \frac{\left\langle \zeta_6^{(4)}, \tilde{Y}_5 \right\rangle}{\| \tilde{Y}_5 \|^2};$$

$$\beta_1 = -\left\langle \zeta_0, \Psi_1 \right\rangle; \beta_2 = -\left\langle \zeta_0, \Psi_2 \right\rangle;$$

$$\beta_3 = -\left\langle \zeta_0, \Psi_3 \right\rangle; \beta_4 = -\left\langle \zeta_0, \Psi_4 \right\rangle;$$

$$\beta_5 = -\left\langle \zeta_0, \Psi_5 \right\rangle; \beta_6 = -\left\langle \zeta_0, \Psi_6 \right\rangle.$$

The notations introduced are similar to those in Appendix 1, with the exception that not only the functions $\zeta_k(\lambda r, \phi)$ but also $\zeta_k^{(i)}(\lambda r, \phi)$ are present in the corresponding scalar products.

With the above notations, the coefficients a_k take the following form:

$$a_6 = \frac{\beta_6}{\| \tilde{Y}_6 \|}; a_5 = \frac{\beta_5}{\| \tilde{Y}_5 \|} - \frac{\beta_6 \rho_5}{\| \tilde{Y}_6 \|};$$

$$a_4 = \frac{\beta_4}{\| \tilde{Y}_4 \|} - \frac{\beta_5 \omega_4}{\| \tilde{Y}_5 \|} + \frac{-\beta_6 \rho_4 + \beta_6 \rho_5 \omega_4}{\| \tilde{Y}_6 \|};$$

$$a_3 = \frac{\beta_3}{\| \tilde{Y}_3 \|} - \frac{\beta_4 \delta_3}{\| \tilde{Y}_4 \|} + \frac{-\beta_5 \omega_3 + \beta_5 \omega_4 \delta_3}{\| \tilde{Y}_5 \|} + (-\beta_6 \rho_3 + \beta_6 \rho_4 \delta_3 + \beta_6 \rho_5 \omega_3 - \beta_6 \rho_5 \omega_4 \delta_3) / \| \tilde{Y}_5 \|;$$

$$a_2 = \frac{\beta_2}{\| \tilde{Y}_2 \|} - \frac{\beta_3 \gamma_2}{\| \tilde{Y}_3 \|} + \frac{-\beta_4 \delta_2 + \beta_4 \delta_3 \gamma_2}{\| \tilde{Y}_4 \|} + (-\beta_5 \omega_2 + \beta_5 \omega_3 \gamma_2 + \beta_5 \omega_4 \delta_2 - \beta_5 \omega_4 \delta_3 \gamma_2) / \| \tilde{Y}_5 \| + (-\beta_6 \rho_2 + \beta_6 \rho_3 \gamma_2 + \beta_6 \rho_4 \delta_2 - \beta_6 \rho_4 \delta_3 \gamma_2 + \beta_6 \rho_5 \omega_2 - \beta_6 \rho_5 \omega_3 \gamma_2 - \beta_6 \rho_5 \omega_4 \delta_2 + \beta_6 \rho_5 \omega_4 \delta_3 \gamma_2) / \| \tilde{Y}_6 \|;$$

$$a_1 = \frac{\beta_1}{\| \tilde{Y}_1 \|} - \frac{\beta_2 \varepsilon}{\| \tilde{Y}_2 \|} + \frac{-\beta_3 \gamma_1 + \beta_3 \gamma_2 \varepsilon}{\| \tilde{Y}_3 \|} + (-\beta_4 \delta_1 + \beta_4 \delta_2 \varepsilon + \beta_4 \delta_3 \gamma_1 - \beta_4 \delta_3 \gamma_2 \varepsilon) / \| \tilde{Y}_4 \| + (-\beta_5 \omega_1 + \beta_5 \omega_2 \varepsilon + \beta_5 \omega_3 \gamma_1 - \beta_5 \omega_3 \gamma_2 \varepsilon + \beta_5 \omega_4 \delta_1 - \beta_5 \omega_4 \delta_2 \varepsilon - \beta_5 \omega_4 \delta_3 \gamma_1 + \beta_5 \omega_4 \delta_3 \gamma_2 \varepsilon) / \| \tilde{Y}_5 \| + (-\beta_6 \rho_1 + \beta_6 \rho_2 \varepsilon + \beta_6 \rho_3 \gamma_1 - \beta_6 \rho_3 \gamma_2 \varepsilon + \beta_6 \rho_4 \delta_1 - \beta_6 \rho_4 \delta_2 \varepsilon - \beta_6 \rho_4 \delta_3 \gamma_1 + \beta_6 \rho_4 \delta_3 \gamma_2 \varepsilon + \beta_6 \rho_5 \omega_1 - \beta_6 \rho_5 \omega_2 \varepsilon - \beta_6 \rho_5 \omega_3 \gamma_1 + \beta_6 \rho_5 \omega_3 \gamma_2 \varepsilon - \beta_6 \rho_5 \omega_4 \delta_1 + \beta_6 \rho_5 \omega_4 \delta_2 \varepsilon + \beta_6 \rho_5 \omega_4 \delta_3 \gamma_1 - \beta_6 \rho_5 \omega_4 \delta_3 \gamma_2 \varepsilon) / \| \tilde{Y}_6 \|.$$

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