DIRECT AND INVERSE PROBLEMS FOR A WAVE EQUATION WITH DISCONTINUOUS COEFFICIENTS

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The present article is devoted to the studies in solutions of partial differential equations with discontinuous coefficients for the highest derivatives. This line of investigation is not only of purely academic interest for mathematicians, but plays an important part in the theory of sounding of unknown media composed of various substances. The direct and inverse problems have been considered. The theorem of existence and of the solution-uniqueness was proved for the first of them. For inverse problems, the uniqueness of the solution was proved.

The integro-differential equation, which is a consequence of the physical laws, was used for solving the direct problem in the derivation of formulae. The meaning of the inverse problems lies in determination of a junction point of different materials and a wave velocity. The used nature of the proof allows us to construct an appropriate numerical algorithm.

Key words: differential equation; discontinuous coefficient; sounding of unknown media; direct and inverse problems

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Introduction

To date, partial differential equations with discontinuous coefficients have not been sufficiently studied. Yet, a number of works have been dedicated to this subject [1 – 12].

In this study, we consider a half-plane

$$R_2^+ = ((x,t), -\infty < x < \infty, t > 0)$$

with the Cauchy problem:

$$\alpha(x)\frac{\partial^2 U(x,t)}{\partial t^2} - \beta(x)\frac{\partial^2 U(x,t)}{\partial x^2} = f(x,t), \quad (x,t) \in R_2^+,\ \alpha(x),\beta(x) > 0, \tag{1}$$

$$U(x,0) = \varphi(x),\ U_t(x,0) = \psi(x). \tag{2}$$

We assume that the function $\varphi(x)$ has continuous derivatives up to and including the second order, while $\psi(x)$ has a continuous first derivative. The function $f(x,t)$ has continuous first-order partial derivatives with $(x,t) \in \overline{R_2^+}$.

The direct problem (1), (2) consists in finding $U(x,t)$ for the given functions $\alpha(x), \beta(x), \varphi(x), \psi(x), f(x,t)$. Its physical meaning is in describing the process of transverse vibrations of a string or longitudinal vibrations of a rod.

The solution of this problem for constant values of α and β is well-known and is represented by d'Alembert's formula. We consider the case of discontinuous coefficients $\alpha(x), \beta(x)$, which corresponds to a string or a rod composed of different materials.

In the particular case when $f = 0$, our results for the direct problem are similar to those presented in [8, 9]. Refs. [10 – 12] also contain various generalizations of d'Alembert's formula. In our study, we use only [10, pp. 75 – 77], where a problem of type (1), (2) is written in the form of an integro-differential equation for a generalized solution in the class of piecewise smooth functions. As for the inverse problems described in this paper, so far we have been unable to uncover similar results by other authors.

While this study contains a substantial amount of cumbersome calculations, we formulate only the fundamental ones in detail, and confine ourselves to outlining the corresponding analysis schemes for other similar actions.

Notations, definitions and direct problem statement

In Eq. (1), functions $\alpha(x),\beta(x)$ are assumed to be piecewise constant, that is,

$$\alpha(x)=\alpha_1,\beta(x)=\beta_1,\ \ x<x_0;$$

$$\alpha(x)=\alpha_2,\beta(x)=\beta_2,\ \ x\ge x_0,$$

where x_0 is a fixed number; α_i,β_i $i=1,2$, are positive numbers.

We will use the following notations below:

$$\gamma_i=\sqrt{\alpha_i\beta_i},\ \ a_i=\sqrt{\beta_i}\,/\,\sqrt{\alpha_i},$$

$$i=1,2,\ a(x)=\sqrt{\beta(x)}\,/\,\sqrt{\alpha(x)}.$$

The notation $\partial_1\omega(x,t)$, $\partial_2\omega(x,t)$ will be used for first derivatives of an arbitrary function $\omega(x,t)$ with respect to x and t in addition to the traditional notation. The left-hand side of Eq. (1) is denoted as $LU(x,t)$ in short.

Let us select the following subsets in the half-plane R_2^+: G_1 is the domain between the straight lines $t=0$ and $x=x_0-a_1t$; G_2 is the domain between the straight lines $t=0$ and $x=x_0+a_2t$; G_3 is the domain between the straight lines $x=x_0+a_2t$ and $x=x_0$; G_4 is the domain between the straight lines $x=x_0$ and $x=x_0-a_1t$; $G_0=G_1\cup G_2\cup G_3\cup G_4$.

Let us write the coupling conditions for the boundaries between the domains G_i, $i=1,\dots,4$:

$$\lim_{x\to x_0-0}\partial_2U(x,t)=\lim_{x\to x_0+0}\partial_2U(x,t),\qquad(3)$$

$$\lim_{x\to x_0-0}\beta_1\partial_1U(x,t)=\lim_{x\to x_0+0}\beta_2\partial_1U(x,t),\qquad(4)$$

$$\{\partial_2U(x,t)-a_1\partial_1U(x,t)\}=0,\ \ x=x_0-a_1t,\qquad(5)$$

$$\{\partial_2U(x,t)+a_2\partial_1U(x,t)\}=0,\ \ x=x_0+a_2t.\qquad(6)$$

The braces in conditions (5), (6) and below denote the jumps of the functions at the boundary points, when the limit value for $(x,t)\in G_i$ is obtained by subtracting the limit value of the same function at $(x,t)\in G_j$, $i>j$. Below we shall establish that conditions (3) – (6) are a consequence of Hooke's law and the law of conservation of momentum.

The solution of problem (1), (2) is sought in the class of functions that are continuous in $\overline{R_2^+}$ and piecewise smooth, such that $U(x,t)$ has in G_0 all partial derivatives up to and including the second order, uniformly continuous at the intersection of each domain G_i, $i=1,\dots,4$ with a circle of any radius and a center at the origin. Let us regard a function of this class as a generalized solution of problem (1), (2) if it satisfies Eq. (1) in G_0 and conditions (2) – (6).

Below, we use curvilinear integrals of the second kind only. The notation (PQ) is used for a curve starting at point P and ending at point Q. If the curve is the boundary of a simply-connected bounded domain, the orientation adopted is such that the domain is located on the left for a point moving along the curve.

Let us agree to denote the current points of the domains G_i, $i=1,\dots,4$ as $M_i=(x,t)$ and construct the following scheme. Taking an arbitrary point $M_1=(x,t)\in G_1$ and using the current variables (ξ,τ), we draw through M_1 two straight lines:

$$\xi-x=a_1(\tau-t),$$

$$\xi-x=-a_1(\tau-t).$$

The first line intersects the horizontal axis of the coordinate system at point $A_1=(0,x-a_1t)$, and the second at point $B_1=(0,x+a_2t)$. The triangle with the vertices A_1, M_1, B_1 is denoted as $G(M_1)$. Similarly, for an arbitrary point $M_2=(x,t)\in G_2$ let us use straight lines

$$\xi-x=a_2(\tau-t),$$

$$\xi-x=-a_2(\tau-t)$$

and obtain points

$$A_2=(0,x-a_2t),\ B_2=(0,x+a_2t)$$

and a triangle $G(M_2)$.

Construction is more complicated in domains G_3, G_4. Let $M_3=(x,t)\in G_3$, $X_0=(x_0,0)$ (Fig. 1). Let us consider the straight lines

$$\xi-x=-a_2(\tau-t),$$

$$\xi-x=a_2(\tau-t).$$

The first line intersects the Ox axis at point $B_3=(0,x+a_2t)$ and the second intersects

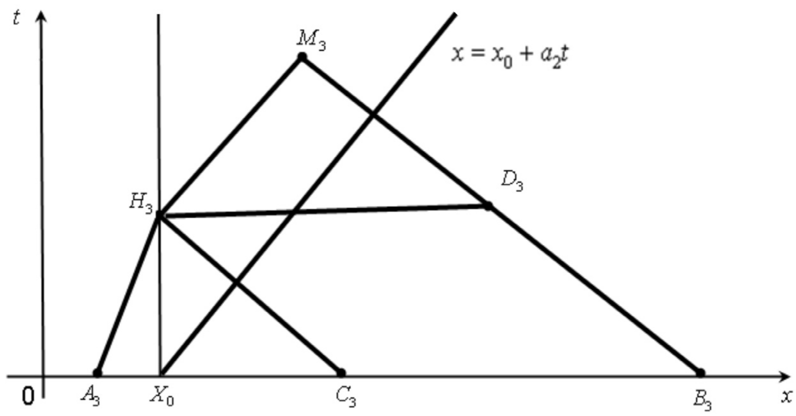

Fig. 1. Illustration to constructing the function $U(x, t)$ for an arbitrary point $M_3 = (x,t) \in G_3$

the line $\xi = x_0$ at point $H_3 = (x_0,h_3)$, where $h_3 = t - (x - x_0)/a_2$. Next, we draw through point H_3 a straight line

$$\xi - x_0 = a_1(\tau - h_3),$$

intersecting the Ox axis at point $A_3 = (0,x_0 - a_1h_3)$.

The quadrangle with the vertices A_3, H_3, M_3, B_3 is denoted as $G(M_3)$. We draw a horizontal line from point H_3, intersecting the line

$$\xi - x = -a_2(\tau - t)$$

at point $D_3 = (2x - x_0, h_3)$.

In addition, we draw a line

$$\xi - x_0 = -a_2(\tau - h_3),$$

intersecting the Ox axis at point $C_3 = (0, x_0 + a_2h_3)$.

The triangle with the vertices A_3, C_3, H_3 is denoted as $G(H_3)$. The construction is similar in domain G_4 for an arbitrary point $M_4 = (x,t) \in G_4$ (Fig. 2). Namely, we use lines

$$\xi - x = a_1(\tau - t),$$

$$\xi - x = -a_1(\tau - t).$$

The first line intersects the Ox axis at point $A_4 = (0, x - a_1t)$ and the second one intersects the line $\xi = x_0$ at point $H_4 = (x_0, h_4)$, where $h_4 = t + (x - x_0)/a_1$.

Next, we draw through point H_4 a straight line

$$\xi - x_0 = -a_2(\tau - h_4),$$

intersecting the Ox axis at point $B_4 = (0, x_0 + a_2h_4)$. The quadrangle with the vertices A_4, M_4, H_4, B_4 is denoted as $G(M_4)$. In addition, we draw a line

$$\xi - x_0 = a_1(\tau - h_4),$$

intersecting the Ox axis at point $C_4 = (0, x_0 - a_1h_4)$. The triangle with the vertices C_4, H_4, B_4 is denoted as $G(H_4)$.

The existence and uniqueness of the solution for the direct problem

Let us denote the values of the function $U(x, t)$ in G_i as $U_i(x, t)$, $i = 1, ..., 4$.

Theorem 1. *Under all the assumptions made, there exists a unique generalized solution of problem (1), (2), represented by formulae*

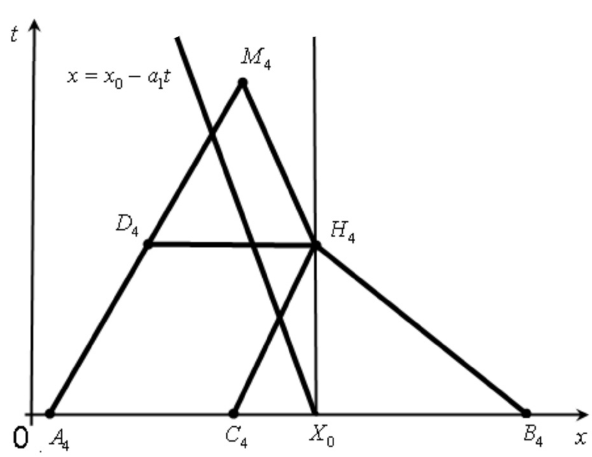

Fig. 2. Illustration to constructing the function $U(x, t)$ for an arbitrary point $M_4 = (x,t) \in G_4$

$$U_i(M_i) = \frac{U(A_i) + U(B_i)}{2} +$$

$$+ \frac{1}{2\gamma_i} \int_{(A_iB_i)} \alpha_i \psi(\xi) d\xi + \frac{1}{2\gamma_i} \int_{G(M_i)} f(\xi, \tau) d\xi d\tau, \quad (7)$$

$$i = 1, 2,$$

$$U_3(M_3) =$$

$$= \frac{1}{2}\left[\frac{2\gamma_1}{\gamma_1 + \gamma_2} U(A_3) + U(B_3) + \frac{\gamma_2 - \gamma_1}{\gamma_1 + \gamma_2} U(C_3) \right] +$$

$$+ \frac{1}{2\gamma_2}\left[\frac{\gamma_2 - \gamma_1}{\gamma_1 + \gamma_2} \int_{(A_3C_3)} \alpha(\xi)\psi(\xi) d\xi + \right. \quad (8)$$

$$\left. + \int_{(A_3B_3)} \alpha(\xi)\psi(\xi) d\xi \right] + \frac{1}{2\gamma_2}\left[\frac{\gamma_2 - \gamma_1}{\gamma_1 + \gamma_2} \times \right.$$

$$\left. \times \int_{G(H_3)} f(\xi, \tau) d\xi d\tau + \int_{G(M_3)} f(\xi, \tau) d\xi d\tau \right];$$

$$U_4(M_4) =$$

$$= \frac{1}{2}\left[\frac{2\gamma_1}{\gamma_1 + \gamma_2} U(A_4) + U(B_4) + \frac{\gamma_1 - \gamma_2}{\gamma_1 + \gamma_2} U(C_4) \right] +$$

$$+ \frac{1}{2\gamma_1}\left[\frac{\gamma_1 - \gamma_2}{\gamma_1 + \gamma_2} \int_{(A_4C_4)} \alpha(\xi)\psi(\xi) d\xi + \right. \quad (9)$$

$$\left. + \int_{(A_4B_4)} \alpha(\xi)\psi(\xi) d\xi \right] + \frac{1}{2\gamma_1}\left[\frac{\gamma_1 - \gamma_2}{\gamma_1 + \gamma_2} \times \right.$$

$$\left. \times \int_{G(H_4)} f(\xi, \tau) d\xi d\tau + \int_{G(M_4)} f(\xi, \tau) d\xi d\tau \right].$$

Proof. The theorem is proved in three stages

1. *Obtaining formulae* (*7*) – (*9*). We should note that the solution in domains G_1 and G_2 is given by d'Alembert's classical formula. Now let us focus on domain G_3. A problem coinciding in meaning with problem (1), (2) was considered in [10, pp. 75 – 77], but under different restrictions. Namely, an integro-differential equation with respect to the function $u(x,t)$ was studied instead of Eq. (1):

$$\int_{\partial G} \alpha(\xi) \partial_2 u(\xi, \tau) d\xi + \beta(\xi) \partial_1 u(\xi, \tau) d\tau + \quad (10)$$

$$+ \int_G f(\xi, \tau) d\xi d\tau = 0. \quad (10)$$

Additionally, the initial conditions are given:

$$u(x, 0) = \varphi(x), \; u_t(x, 0) = \psi(x). \quad (11)$$

G in equality (10) is an arbitrary bounded simply-connected domain in R_2^+, and its boundary ∂G is a piecewise smooth curve of class C^1. The function $u(x,t)$ is continuous in $\overline{R_2^+}$, and its partial derivatives $\partial_1 u(x, t)$, $\partial_2 u(x, t)$, are piecewise continuous with possible discontinuities of the first kind on certain lines. In this case, discontinuities are allowed within G and it is also possible that the line of discontinuities coincides with a part of ∂G, and then the derivatives $\partial_1 u(\xi, \tau)$, $\partial_2 u(\xi, \tau)$ are substituted in Eq. (10) by their limit values within domain G. We should note that Eq. (10) is a consequence of Hooke's law and the law of conservation of momentum. Accordingly, the conclusions obtained from (10) are also a consequence of these laws.

Let us take an arbitrary point $M_3 = (x,t) \in G_3$ and use Eq. (10) with respect to quadrangle $G(M_3)$ (see Fig. 1). The curvilinear integrals along the lines (M_3H_3), (H_3A_3), (A_3B_3), (B_3M_3) are denoted, respectively, as I_1, I_2, I_3, I_4. Via direct and simple calculations, we obtain the equalities

$$I_1 = \gamma_2(u(H_3) - u(M_3)),$$

$$I_2 = \gamma_1(u(A_3) - u(H_3)),$$

$$I_3 = \int_{(A_3B_3)} \alpha(\xi)\psi(\xi) d\xi, \; I_4 = \gamma_2(u(B_3) - u(M_3)).$$

Additionally, let us use equality (10) for triangle $G(H_3)$. Let us denote the curvilinear integrals along the lines (H_3A_3), (A_3C_3), (C_3H_3) as J_1, J_2, J_3 and, similar to the above, we have:

$$J_1 = \gamma_1(u(A_3) - u(H_3)), \; J_2 = \int_{(A_3C_3)} \alpha(\xi)\psi(\xi) d\xi,$$

$$J_3 = \gamma_2(u(C_3) - u(H_3)).$$

Next, we use equality (10) for $G(M_3)$ and $G(H_3)$, and taking into account the calculations for I_1, I_2, I_3, I_4, J_1, J_2, J_3, we obtain for $u(M_3)$ the expression coinciding with the right-hand side of the equality for $U_3(M_3)$. Identifying $u(x,t)$ with $U_3(x,t)$, we obtain formula (8). $U_4(x,t)$ is obtained in much the same way. In this case, quadrangle $G(M_4)$ and triangle $G(H_4)$ are used in equality (10).

2. *Existence of a solution for the direct problem.* Let us calculate the limits for $U(x,t)$ with the arguments tending to points on the lines

$$x - x_0 = a_2 t,\ x - x_0 = -a_1 t,\ x = x_0.$$

If we use the expressions for A_i, B_i, C_i, H_i, $i = 1,..., 4$, via (x,t), it is evident that the limits of the functions $U_i(x,t)$ coincide at these points, i.e., the function $U(x,t)$ represented by formulae (7) – (9) is continuous in $\overline{R_2^+}$.

Now let us verify that $U_i(x,t)$ satisfy Eq. (1) in G_i, $i = 1,..., 4$. We should note that this does not have to be verified for functions $U_1(x,t)$ and $U_2(x,t)$, as they are represented by d'Alembert's classical formulae, which also means that conditions (2) are satisfied for $U(x,t)$. It is convenient for us to examine separate parts of formula (8),

$$U_3(x,t) = U_{3,\varphi}(x,t) + U_{3,\psi}(x,t) + U_{3,f}(x,t),$$

where

$$U_{3,\varphi}(x,t) = \frac{1}{2}\left[\frac{2\gamma_1}{\gamma_1+\gamma_2}\varphi(x_0 - a_1 h_3) + \varphi(x + a_2 t) + \frac{\gamma_2-\gamma_1}{\gamma_1+\gamma_2}\varphi(x_0 + a_2 h_3)\right],$$

$$h_3 = t - (x - x_0)/a_2,\ (x,t) \in G_3; \quad (12)$$

$$U_{3,\psi}(x,t) = \frac{1}{2\gamma_2}\left[\frac{\gamma_2-\gamma_1}{\gamma_1+\gamma_2}\int_{(A_3C_3)}\alpha(\xi)\psi(\xi)d\xi + \int_{(A_3B_3)}\alpha(\xi)\psi(\xi)d\xi\right]; \quad (13)$$

$$U_{3,f}(x,t) = \frac{1}{2\gamma_2}\left[\frac{\gamma_2-\gamma_1}{\gamma_1+\gamma_2}\int_{G(H_3)} f(\xi,\tau)d\xi d\tau + \int_{G(M_3)} f(\xi,\tau)d\xi d\tau\right]. \quad (14)$$

Let us calculate the derivatives of the function $U_{3,\varphi}(x,t)$:

$$\frac{\partial U_{3,\varphi}(x,t)}{\partial t} = \frac{1}{2}\left[\frac{-2\gamma_1 a_1}{\gamma_1+\gamma_2}\varphi'(x_0 - a_1 h_3) + a_2\varphi'(x + a_2 t) + \frac{\gamma_2-\gamma_1}{\gamma_1+\gamma_2}a_2\varphi'(x_0 + a_2 h_3)\right], \quad (15)$$

$$\frac{\partial^2 U_{3,\varphi}(x,t)}{\partial t^2} = \frac{1}{2}\left[\frac{2\gamma_1 a_1^2}{\gamma_1+\gamma_2}\varphi''(x_0 - a_1 h_3) + a_2^2\varphi''(x + a_2 t) + \frac{\gamma_2-\gamma_1}{\gamma_1+\gamma_2}a_2^2\varphi''(x_0 + a_2 h_3)\right]; \quad (16)$$

$$\frac{\partial U_{3,\varphi}(x,t)}{\partial x} = \frac{1}{2}\left[\frac{2\gamma_1 a_1}{(\gamma_1+\gamma_2)a_2}\varphi'(x_0 - a_1 h_3) + \varphi'(x + a_2 t) - \frac{\gamma_2-\gamma_1}{\gamma_1+\gamma_2}\varphi'(x_0 + a_2 h_3)\right]; \quad (17)$$

$$\frac{\partial^2 U_{3,\varphi}(x,t)}{\partial x^2} = \frac{1}{2}\left[\frac{2\gamma_1 a_1^2}{(\gamma_1+\gamma_2)a_2^2}\varphi''(x_0 - a_1 h_3) + \varphi''(x + a_2 t) + \frac{\gamma_2-\gamma_1}{\gamma_1+\gamma_2}\varphi''(x_0 + a_2 h_3)\right]. \quad (18)$$

Using equalities (16), (18), we write

$$LU_{3,\varphi}(x,t) = \varphi''(x_0 - a_1 h_3)\left[\frac{\gamma_1 a_1^2\alpha_2}{\gamma_1+\gamma_2} - \frac{\gamma_1 a_1^2\beta_2}{(\gamma_1+\gamma_2)a_2^2}\right] + \varphi''(x + a_2 t)\left[\frac{a_2^2\alpha_2}{2} - \frac{\beta_2}{2}\right] + \varphi''(x_0 + a_2 h_3)\left[\frac{a_2^2\alpha_2(\gamma_2-\gamma_1)}{2(\gamma_1+\gamma_2)} - \frac{\beta_2(\gamma_2-\gamma_1)}{2(\gamma_1+\gamma_2)}\right].$$

It is easy to verify that all the expressions in square brackets on the right-hand side of the last equality are equal to zero, i.e., $LU_{3,\varphi}(x,t) = 0$, $(x,t) \in G_3$.

Next, let us calculate the partial derivatives of the function $U_{3,\psi}(x,t)$:

$$\frac{\partial U_{3,\psi}(x,t)}{\partial t}=\frac{1}{2\gamma_2}\left[\frac{\gamma_2-\gamma_1}{\gamma_1+\gamma_2}(\alpha_1 a_1\psi(x_0-a_1h_3)+\right.$$

$$+\alpha_2 a_2\psi(x_0+a_2h_3))+\alpha_1 a_1\psi(x_0-a_1h_3)+ \qquad (19)$$

$$\left.+\alpha_2 a_2\psi(x+a_2t)\right];$$

$$\frac{\partial^2 U_{3,\psi}(x,t)}{\partial t^2}=\frac{1}{2\gamma_2}\left[\frac{\gamma_2-\gamma_1}{\gamma_1+\gamma_2}(-\alpha_1 a_1^2\psi'(x_0-a_1h_3)+\right.$$

$$\left.+\alpha_2 a_2^2\psi'(x_0+a_2h_3))\right]+ \qquad (20)$$

$$+\frac{1}{2\gamma_2}[-\alpha_1 a_1^2\psi'(x_0-a_1h_3)+\alpha_2 a_2^2\psi'(x+a_2t)],$$

$$\frac{\partial U_{3,\psi}(x,t)}{\partial x}=\frac{1}{2\gamma_2}\left[\frac{\gamma_2-\gamma_1}{\gamma_1+\gamma_2}\left(-\frac{\alpha_1 a_1}{a_2}\psi(x_0-a_1h_3)-\right.\right.$$

$$\left.-\alpha_2\psi(x_0+a_2h_3)\right)-\frac{\alpha_1 a_1}{a_2}\psi(x_0-a_1h_3)+ \qquad (21)$$

$$\left.+\alpha_2\psi(x+a_2t)\right];$$

$$\frac{\partial^2 U_{3,\psi}(x,t)}{\partial x^2}=\frac{1}{2\gamma_2}\left[\frac{\gamma_2-\gamma_1}{\gamma_1+\gamma_2}\times\right.$$

$$\left.\times\left(-\alpha_1\frac{a_1^2}{a_2^2}\psi'(x_0-a_1h_3)+\alpha_2\psi'(x_0+a_2h_3)\right)\right]+ \qquad (22)$$

$$+\frac{1}{2\gamma_2}\left[-\alpha_1\frac{a_1^2}{a_2^2}\psi'(x_0-a_1h_3)+\alpha_2\psi'(x+a_2t)\right].$$

It follows from equalities (20), (22) that $LU_{3,\psi}(x,t)=0$. Now let us focus on the expression $U_{3,f}(x,t)$. Firstly, let us consider the integral $F(x,t)$, representing it as a sum of two terms:

$$F(x,t)=F_1(x,t)+F_2(x,t),$$

where $F_1(x,t)$ is the integral with respect to $f(\xi,\tau)$ over a triangle with the vertices H_3, M_3, D_3, and $F_2(x,t)$ is the integral with respect to the same function over the remaining part of $G(M_3)$. Consequently,

$$F_1(x,t)=\int_{h_3}^{t}\int_{x-a_2(t-\tau)}^{x+a_2(t-\tau)} f(\xi,\tau)d\xi d\tau, \qquad (23)$$

$$F_2(x,t)=\int_{t}^{h_3}\int_{x-a_1(h_3-\tau)}^{x+a_2(t-\tau)} f(\xi,\tau)d\xi d\tau. \qquad (24)$$

Let us calculate the partial derivatives of $F_1(x,t)$ and $F_2(x,t)$:

$$\frac{\partial F_1(x,t)}{\partial t}=-\int_{x-a_2(t-h_3)}^{x+a_2(t-h_3)} f(\xi,h_3)d\xi+$$

$$+\int_{h_3}^{t}[a_2 f(x+a_2(t-\tau),\tau)+ \qquad (25)$$

$$+a_2 f(x-a_2(t-\tau),\tau)]d\tau;$$

$$\frac{\partial^2 F_1(x,t)}{\partial t^2}=-\int_{x_0}^{2x-x_0}\partial_2 f(\xi,h_3)d\xi+2a_2 f(x,t)-$$

$$-a_2 f(2x-x_0,h_3)-a_2 f(x_0,h_3)+ \qquad (26)$$

$$+\int_{h_3}^{t}a_2^2[\partial_1 f(x+a_2(t-\tau),\tau)-$$

$$-\partial_1 f(x-a_2(t-\tau),\tau)]d\tau;$$

$$\frac{\partial F_2(x,t)}{\partial t}=\int_{x_0}^{2x-x_0} f(\xi,h_3)d\xi+$$

$$+\int_{0}^{h_3}[a_2 f(x+a_2(t-\tau),\tau)+ \qquad (27)$$

$$+a_1 f(x_0-a_1(h_3-\tau),\tau)]d\tau;$$

$$\frac{\partial^2 F_2(x,t)}{\partial t^2}=$$

$$=-\int_{x_0}^{2x-x_0}\partial_2 f(\xi,h_3)d\xi+a_2 f(2x-x_0,h_3)+$$

$$+a_1 f(x_0,h_3)+\int_{0}^{h_3}[a_2^2\partial_1 f(x+a_2(t-\tau),\tau)-$$

$$-a_1^2\partial_1 f(x-a_1(t-\tau),\tau)]d\tau.$$

It follows from the last equality and formula (26) that

$$\frac{\partial^2 F(x,t)}{\partial t^2}=2a_2 f(x,t)+(a_1-a_2)f(x_0,h_3)+$$

$$+\int_{h_3}^{t}a_2^2[\partial_1 f(x+a_2(t-\tau),\tau)-$$

$$-\partial_1 f(x-a_2(t-\tau),\tau)]d\tau+$$

$$+\int_0^{h_3}[a_2^2\partial_1 f(x+a_2(t-\tau),\tau)-$$
$$-a_1^2\partial_1 f(x-a_1(h_3-\tau),\tau)]d\tau.$$

By similar direct calculations we obtain the equality

$$\frac{\partial^2 F(x,t)}{\partial x^2}=\left[\frac{a_1}{a_2^2}-\frac{1}{a_2}\right]f(x_0,h_3)+$$
$$+\int_{h_3}^{t}[\partial_1 f(x+a_2(t-\tau),\tau)-$$
$$-\partial_1 f(x-a_2(t-\tau),\tau)]d\tau+$$
$$+\int_0^{h_3}\left[a_2^2\partial_1 f(x+a_2(t-\tau),\tau)-\right.$$
$$\left.-\frac{a_1^2}{a_2^2}\partial_1 f(x-a_1(h_3-\tau),\tau)\right]d\tau.$$

Using the formulae obtained for the second derivatives of the function $F(x,t)$, we verify that $LF(x,t)=f(x,t)$.

Now let us consider the integral $\Phi(x,t)$ with respect to $f(\xi,\tau)$ over triangle $G(H_3)$, i.e.,

$$\Phi(x,t)=\int_0^{h_3}\int_{x_0-a_1(h_3-\tau)}^{x_0+a_2(h_3-\tau)} f(\xi,\tau)d\xi d\tau.$$

Let us calculate the derivatives of the function $\Phi(x,t)$:

$$\frac{\partial\Phi(x,t)}{\partial t}=\int_0^{h_3}[a_2 f(x_0+a_2(h_3-\tau),\tau)+$$
$$+a_1 f(x_0-a_1(h_3-\tau),\tau)]d\tau; \tag{28}$$
$$\frac{\partial^2\Phi(x,t)}{\partial t^2}=(a_1+a_2)f(x_0,h_3)+$$
$$+\int_0^{h_3}[a_2^2\partial_1 f(x_0+a_2(h_3-\tau),\tau)-$$
$$-a_1^2\partial_1 f(x_0-a_1(h_3-\tau),\tau)]d\tau;$$
$$\frac{\partial^2\Phi(x,t)}{\partial x^2}=\left(\frac{a_1}{a_2^2}+\frac{1}{a_2}\right)f(x_0,h_3)+$$
$$+\int_0^{h_3}\left[\partial_1 f(x_0+a_2(h_3-\tau),\tau)-\right.$$
$$\left.-\frac{a_1^2}{a_2^2}\partial_1 f(x_0-a_1(h_3-\tau),\tau)\right]d\tau.$$

If follows from here that $L\Phi(x,t)=0,\ (x,t)\in G_3$.

Combining the results of the calculations, we obtain the equality

$$LU_{3,f}(x,t)=f(x,t),\ \ (x,t)\in G_3.$$

In view of the equations

$$LU_{3,\varphi}(x,t)=0,\ \ LU_{3,\psi}(x,t)=0,$$

which we have obtained earlier, we now obtain $LU_3(x,t)=f(x,t)$.

We similarly obtain the equality

$$LU_4(x,t)=f(x,t),\ \ (x,t)\in G_4.$$

The same conclusion can be reached by replacing the variable $x'=2x_0-x$ and using the results we have already obtained. Thus, we have proved that the function $U(x,t)$ satisfies Eq. (1) in G_0 and conditions (2).

It is easy to derive properties (3) – (6) from the equalities obtained for $\partial_1 U(x,t)$ and $\partial_2 U(x,t)$. Therefore, the function $U(x,t)$ represented by formulae (7) – (9) is a generalized solution of problem (1), (2).

3. *Uniqueness of the solution of the direct problem.* To prove this uniqueness, we take two solutions of problem (1), (2) and denote their difference as $V(x,t)$. Let us consider the functions

$$v_1(x,t)=\partial_2 V(x,t)+a(x)\partial_1 V(x,t),$$
$$v_2(x,t)=\partial_2 V(x,t)-a(x)\partial_1 V(x,t).$$

It is easy to verify that the following equalities are satisfied:

$$\partial_2 v_1(x,t)-a(x)\partial_1 v_1(x,t)=0,$$
$$\partial_2 v_2(x,t)+a(x)\partial_1 v_2(x,t)=0, \tag{29}$$
$$v_i(x,0)=0,\ i=1,2\ ,\ (x,t)\in G_0.$$

Let us agree to denote $v_1(x,t), v_2(x,t), V(x,t)$ for $x<x_0$ as $v_1^-(x,t), v_2^-(x,t), V^-(x,t)$, and as $v_1^+(x,t), v_2^+(x,t), V^+(x,t)$ for $x\ge x_0$.

It follows from conditions (5), (6) that the functions $v_1^+(x,t), v_2^-(x,t)$ are continuous. Therefore, we obtain from system (29)

$$v_1^+(x,t)=v_2^-(x,t)=0.$$

For an arbitrary point H on the half-axis $(\underline{x}_0,\underline{t}),\underline{t}>0$, the following equalities are satisfied:

$$v_1^+(H) = \partial_2 V^+(H) + a_2\partial_1 V^+(H) = 0,$$

$$v_2^-(H) = \partial_2 V^-(H) - a_1\partial_1 V^-(H) = 0.$$

Then, since $\partial_2 v_2(x,t)$ is continuous with $x = x_0$ and taking into account condition (4), it follows that $v_1(x,t)$ and $v_2(x,t)$ $v_1(x,t), v_2(x,t)$ are continuous at point H. Thus, we obtain the equalities

$$v_1(x,t) = v_2(x,t) = 0,$$

$$\partial_1 V(x,t) = 0, \partial_2 V(x,t) = 0,$$

where $V(x,t)$ = const.

Therefore, due to the condition $V(x,0) = 0$, we have $V(x,t) = 0$, $(x,t) \in R_2^+$, which actually means that the solution of the direct problem is unique.

Theorem 1 is proved.

Statement and investigation of inverse problems

Two inverse problems are considered in the study.

Problem 1. With the given solution of the direct problem $U(x,t)$ on rays (x_1,t), (x_2,t), (x_3,t), t > 0, where the fixed points x_i, $i = 1,2,3$, satisfy the inequalities $x_1 < x_0$, x_2, $x_3 > x_0$, find the values x_0, a_1, a_2.

Problem 2. *With the given values of, $i = 2,3$ and fixed points x_2, $x_3 > x_0$, find x_0, a_2.*

Both problems have applications in theory on sounding unknown media and determining some of their parameters. In this case, the junction point of different materials (x_0) and the wave propagation velocities (a_i) are unknown. The second problem differs from the first by the smaller amount of the known data and, correspondingly, smaller amount of the data to be determined.

Theorem 2. *Each of the inverse problems 1, 2 has at most one solution if the following condition is satisfied:*

$$\varphi'(x_0)(\beta_1 - \beta_2) \neq 0. \tag{30}$$

Proof. First of all, let us analyze the derivative $\partial_2 U(x,t)$ with a fixed x, $x \neq x_0$, $t > 0$. Let us assume to be definite that $x > x_0$. Then the ray $(\underline{x},t), t > 0$, lies in domains G_2 and G_3, intersecting the boundary between them at the point

$$P = (x, (x - x_0) / a_2).$$

We have already carried out an analysis of $\partial_2 U(x,t)$ for $(x,t) \in G_3$ in proving Theorem 1, and the results are represented by formulae (15), (19), (25), (27).

Now let us carry out a similar analysis of $\partial_2 U(x,t)$, $(x,t) \in G_2$. Using equality (7) for $i = 2$, let us represent $U_2(x,t)$ as a sum

$$U_2(x,t) = U_{2,\varphi}(x,t) + U_{2,\psi}(x,t) + U_{2,f}(x,t),$$

where

$$U_{2,\varphi}(x,t) = \frac{\varphi(x - a_2 t) + \varphi(x + a_2 t)}{2};$$

$$U_{2,\psi}(x,t) = \frac{1}{2\gamma_2}\int_{x-a_2 t}^{x+a_2 t} \alpha_2 \psi(\xi) d\xi;$$

$$U_{2,f}(x,t) = \frac{1}{2\gamma_2}\int_0^t \int_{x-a_2(t-\tau)}^{x+a_2(t-\tau)} f(\xi,\tau) d\xi d\tau.$$

Calculating the derivatives of these functions with respect to t and using the limit $(x,t) \to P$, we obtain:

$$\frac{\partial U_{2,\varphi}(P)}{\partial t} = \frac{a_2}{2}(\varphi'(2x - x_0) - \varphi'(x_0)); \tag{31}$$

$$\frac{\partial U_{2,\psi}(P)}{\partial t} = \frac{1}{2}(\psi(2x - x_0) + \psi(x_0)); \tag{32}$$

$$\frac{\partial U_{2,f}(P)}{\partial t} = \frac{a_2}{2\gamma_2}\int_0^{(x-x_0)/a_2} [f(2x - x_0 - a_2\tau, \tau) + f(x_0 + a_2\tau, \tau)] d\tau. \tag{33}$$

Using equalities (25), (27), (28), we obtain the limit value of $\partial_2 U_{3,f}(x,t)$ at point P:

$$\frac{\partial U_{3,f}(P)}{\partial t} = \frac{a_2}{2\gamma_2}\int_0^{(x-x_0)/a_2} [f(2x - x_0 - a_2\tau, \tau) + f(x_0 + a_2\tau, \tau)] d\tau. \tag{34}$$

Using limit $(x,t) \to P$, $(x,t) \in G_3$ in equalities (15), (19), (25), (27), (28) and comparing the obtained expressions in the right-hand sides of equalities (15) and (31), (19) and (32), (33) and (34), we reach the following conclusion:

$$\partial_2 U_{3,\psi}(P) = \partial_2 U_{2,\psi}(P),$$

$$\partial_2 U_{3,f}(P) = \partial_2 U_{2,f}(P);$$

$$\partial_2 U_{3,\varphi}(P) - \partial_2 U_{2,\varphi}(P) = = \varphi'(x_0)(\beta_2 - \beta_1) / (\gamma_2 + \gamma_1.)$$

Thus, the terms of the function $U(x,t)$ that contain $\psi(x)$, and $f(x,t)$ have continuous derivatives with respect to t. The term containing the function $\varphi(x)$ has a discontinuous derivative with respect to t at point P, i.e.,

$$\left\{\frac{\partial U}{\partial t}(P)\right\} = \varphi'(x_0)\frac{\beta_2 - \beta_1}{\gamma_2 + \gamma_1}. \quad (35)$$

In a similar manner, we can verify for the case $x < x_0$ that the following equality is satisfied:

$$\left\{\frac{\partial U}{\partial t}(Q)\right\} = \varphi'(x_0)\frac{\beta_1 - \beta_2}{\gamma_2 + \gamma_1}, \quad (36)$$

where Q is the point of intersection of the ray (x,t), $t > 0$ and the straight line $\xi - x_0 = -a_1\tau$.

We should note that, as in Theorem 1, we can replace the variable $x' = 2x_0 - x$ and derive equality (36) from equality (35).

Let us denote the intersection points of the straight line $\xi - x_0 = -a_2\tau$ and the rays (x_2,t), (x_3,t), $t > 0$ as P_2, P_3. It follows from property (35) that $\partial_2 U(x_2,t)$, $\partial_2 U(x_3,t)$ have discontinuities only if $(x_2,t) = P_2$, $(x_3,t) = P_3$, respectively. Thus, points P_2, P_3 are uniquely determined from the data of the problem, and consequently, x_0 and a_2 are uniquely determined as well, which means that the solution of problem 2 is unique.

Using the representation $U(x_1,t)$, $t > 0$ and equality (36), we verify that $\partial_2 U(x_1,t)$ has a discontinuity only at point P_1, which is the intersection of the line $\xi - x_0 = -a_1\tau$ and the ray (x_1,t), $t > 0$. Since P_1 and x_0 are uniquely determined, it follows that a_0 is uniquely determined as well.

Thus, Theorem 2 is proved.

We should bear in mind that if condition (30) is satisfied, then, using the proof of Theorem 2, it is easy to construct the corresponding algorithms.

Note. Since the function $\varphi(x)$ is not given in inverse problems but the inequality $\varphi'(x_0) \neq 0$ has to hold true, Theorem 2 is conditional. To give this theorem a constructive form, it is sufficient to postulate the inhomogeneity of the medium $(\beta_1 \neq \beta_2)$ and additionally set the function $\varphi(x)$.

Conclusion

In this study, we have considered a one-dimensional wave equation describing the transverse vibrations of an inhomogeneous string or longitudinal vibrations of an inhomogeneous rod. We have formulated a direct problem on determining the vibration function in the general case, when the initial state, the initial velocity, and the external force are known and sufficiently arbitrary. We have proved that the solution for this problem exists and is unique, and provided simple and explicit formulae for this solution. In addition, we have considered two inverse problems on finding the junction point of different materials and wave propagation velocities. We have proved that inverse problems have a unique solution if a certain inequality is satisfied.

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