DETERMINING THE CONTACT FORCE OF AN AXIAL COLLISION OF AN ELASTIC ROD WITH A RIGID IMPACTOR

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The problem of axial impact of a rigid body on elastic rod is considered. The Semi-Analytical Method (SEM) and Finite Element Method (FEM) are applied to handle the problem. The SEM of solving the problem implies the quasi-static Hertz theory and numerical integration of obtained differential equations. The number of necessary degrees of freedom of the FEM solution is determined and numerical simulation is carried out. The time of contact interaction and dependence of the contact force on the contact time are calculated. The longitudinal wave propagation in the rod is investigated. The obtained results are compared with the data from natural experiments. An inverse dependence between impacting mass and the accuracy of both methods is discussed. The results of comparison confirm the appropriateness of both methods for solving the problem.

Key words: axial collision; contact force; finite element method

Citation: A.K. Belyaev, A.O. Shurpatov, Ch.-Ch. Ma, Determining the contact force of an axial collision of an elastic rod with a rigid impactor, St. Petersburg Polytechnical State University Journal. Physics and Mathematics. 11 (1) (2018) 102 – 108. DOI: 10.18721/JPM.11113

Introduction

Solving problems of dynamics and stability of thin rods under longitudinal impact requires knowledge of the form and amplitude of the force in the contact zone [1 – 3]. This paper is dedicated to determining the contact force in the axial collision of a rod and an impactor using three fundamentally different approaches: the semi-analytical method (mathematical modeling), the finite element method, and an experiment. Comparing the results obtained by these approaches is particularly interesting, as this allows to assess whether each approach is correct and can be used in the future.

Problem statement

An elastic rod with a length l is considered; one of the rod's ends is fixed (displacements and rotations are forbidden for all points of the cross-section). An impactor of mass m approaches the free end of the rod at the initial time with the velocity V_0, causing a contact interaction in the system (Fig. 1). The gravity forces of the rod and the impactor are not taken into account.

In the general case, the force of elastic contact interaction arises as a result of mutual vibrations of colliding bodies and can be determined from the analysis of their combined dynamic strain.

The goals of the study are to find the time of contact interaction between the rod and the impactor, to construct the dependence of the arising contact force on time, and to experimentally verify the calculations.

Semi-analytical method

Description of the mathematical model. The condition for contact between the bodies

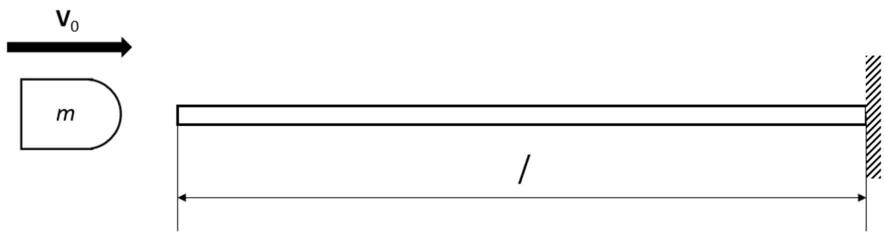

Fig. 1. Schematic for the problem statement (general case):
l is the length of the elastic rod; *m* is the mass of the non-deformable impactor, $\mathbf{V}_0$ is the vector of its initial velocity

is the coincidence of the coordinates of their contact points [4]:

$$v_0 t - \alpha - y_1 - y_2 = 0,$$

where α is the linear convergence of the bodies due to contact deformations; y_1 and y_2 are the dynamic displacements of contact points of both bodies, caused by the contact force $P(t)$ without taking into account local deformations; v_0 is the initial velocity of the impactor.

As the contact area is small, we can neglect its mass. Then we are able to use Hertz's quasi-static contact theory, according to which the contact force P is related to the quantity α by the dependence [5]:

$$P(\alpha) = k\alpha^{3/2},$$

Where k is the coefficient depending on the parameters of the contacting bodies.

In the case of the model under consideration, it has the form [6]:

$$k = \frac{2}{3(1-\mu^2)} E\sqrt{R},$$

where E and μ are Young's modulus and Poisson's ratio, respectively (assuming that the rod and the impactor are made of the same material); R is the radius of the spherical profile of the impactor.

The displacements y_1 and y_2 can be expressed through the contact force, using the reaction of each of the colliding bodies to a unit impulse [7]:

$$y_1 = \int_0^t P(\theta) Y^{(1)}(t-\theta)d\theta,$$

$$y_2 = \int_0^t P(\theta) Y^{(2)}(t-\theta)d\theta,$$

where $Y^{(1)}$, $Y^{(2)}$ are the reactions to the unit impulse of the rod and impactor, respectively; t is the current time, θ is the integration variable; the moment of contact of bodies is taken as the origin.

Substituting these expressions into the contact condition, we obtain an integral equation that determines the contact force:

$$\int_0^t P(\theta) Y(t-\theta)d\theta + [P(t)/k]^{2/3} = v_0 t,$$

where $Y(t) = Y^{(1)}(t) + Y^{(2)}(t)$.

Since the integral term of this equation depends on the values of the contact force at all times θ preceding the one under consideration, with a sufficiently small integration step over the time Δt, we can neglect the change in the force in the integral sum over the interval

$$t - \Delta t \le \theta \le t.$$

In view of the above, the expression for determining the contact force can be written in the following form [8]:

$$P(t) = k[v_0 t - \int_0^{t-\Delta t} P(\theta) Y(t-\theta)d\theta -$$

$$-P(t-\Delta t)\int_0^{\Delta t} Y(\theta)d\theta]^{3/2}.$$

Thus, using a small step Δt with the help of numerical integration, we calculate the dependence of the contact force on time step by step. In this case, for the system under consideration, the reaction of the rod to a unit impulse is as follows [9]:

with $0 < t < \frac{2l}{a}$

$$Y^{(1)}(t) = l^2 / [(2EFa)(at/l)^2];$$

with $\frac{2l}{a} < t < \frac{4l}{a}$

$$Y^{(1)}(t) = l^2 / \{(2EFa)[8 - (4 - at / l)^2]\};$$

with $\frac{4l}{a} < t < \frac{6l}{a}$

$$Y^{(1)}(t) = l^2 / \{(2EFa)[8 + (at / l - 4)^2]\}$$

and so on,

where $a = \sqrt{\frac{E}{\rho}}$ is the speed of sound in the material of the rod.

Since this model does not consider the wave processes occurring in the impactor, its reaction to a unit impulse is determined by the following expression [10]:

$$Y^{(2)}(t) = \frac{t^3}{6m},$$

where m is the impactor mass.

The essence of the semi-analytic method consists in numerical integration of the equation obtained above for calculating the contact force [11].

Results of mathematical modeling. The dependence of the contact force on time was calculated by the semi-analytic method for systems with different parameters. It was obtained from these calculations that the contact force is a smooth time function which has one to three maxima depending on the parameters.

An example of calculating the contact force with three maxima is shown in Fig. 2 (curve *1*) and was obtained with the initial parameters of the system given in Table 1.

The rod and the spherical profile of the impactor were assumed to be made of steel with the characteristics also given in Table 1.

The results obtained were verified by finite element simulation of a system with identical parameters (see Table 1).

Finite-element model

Description of the model. After studying the convergence of the finite element method, we have selected the model shown in Fig. 3. It includes about 300,000 knots and has about 1 million degrees of freedom. Since this problem involves investigating not only the contact interaction of the rod with the impactor but also the wave processes occurring in the rod itself [12], we decided not to condense the grid in the contact area. Thus, a uniform grid was used in the process of finite-element modeling.

The statement of the problem in this model is as follows.

The rod is made of linearly elastic material (as already noted above for the general model)

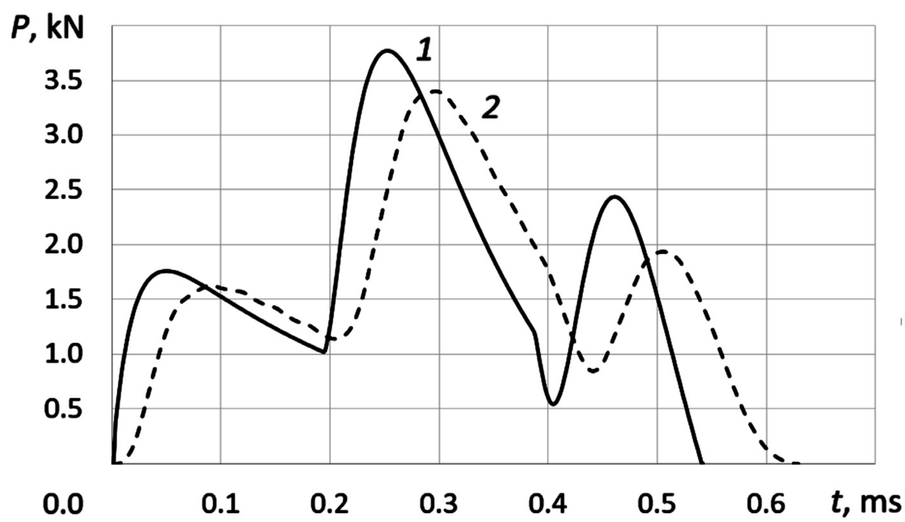

Fig. 2. Dependence of the contact force on time, obtained by the semi-analytic method (curve *1*) and the finite element method (curve *2*); the values of the initial parameters are given in Table 1

Table 1

Initial design and experimental parameters of the system

System element, material	Parameter	Notation	Unit	Value	
				Model	Experiment
Rod	Length	l	m	0.500	0.301
	Cross-section area	S	m^2	$5.0\cdot10^{-5}$	$3.14\cdot10^{-6}$
Impactor	Mass	M	kg	0.5	0.13 – 8.46
	Spherical profile radius	R	m	0.01	1.58 – 6.36
	Initial velocity	V_0	m/s	1.0	0.3225
Steel	Young's modulus	E	N/m^2	$2.1\cdot10^{11}$	$7.342\cdot10^{10}$
	Poisson's coefficient	μ	–	0.30	0.34
	Density	ρ	kg/m^3	7800.0	2696.6

Note: the initial calculated parameters for both models given in the Table are the ones deemed to be the most convenient by the results of the calculation

whose parameters are given in Table 1. One of the rod's ends is fixed, that is, both displacements and rotations are forbidden for all nodes. The opposite end of the rod is free (general model). In this problem, we assumed that gravity was absent, therefore, there is no curvature of the rod at the initial time and the rod is stationary. Unlike the rod, the impactor consists of two materials. Its front part is made of the same material as the rod (it is elastic), and the material of the rear part is perfectly rigid. This composition of the impactor was chosen for two reasons. Firstly, the materials in the contact area have to be identical to correctly compare the results with those obtained by the semi-analytical method [5]. Secondly, this construction minimizes the influence of wave processes in the impactor on the model used [3]. Wave processes cannot occur in a perfectly rigid body, which allows to study only the wave processes that evolve directly in the elastic rod.

Displacement of the nodes of the perfectly rigid part of the impactor is only allowed along the axis of the rod, and all other displacements and rotations are forbidden. All the nodes of the impactor have a velocity directed along the axis to the free end of the rod at the initial time.

As already noted above, one of the goals of the study was to determine the dependence of the contact force on the impact time.

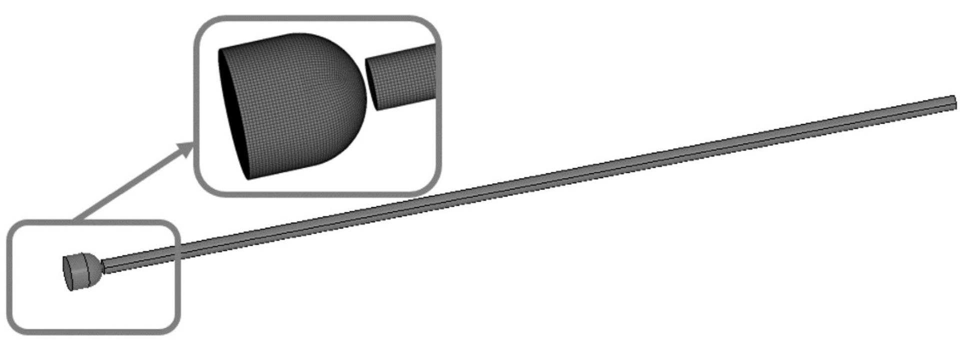

Fig. 3. Schematic of the finite-element model
(an enlarged area of the contact between the rod and the impactor is additionally shown)

Results of finite-element modeling. This problem was modeled by the finite-element method with different parameters of both the rod and the impactor. The most noteworthy results were obtained for the initial parameters identical to those for a semi-analytical modeling method (see Table 1).

The obtained dependence of the contact force on time is shown in Fig. 2 (curve *2*).

Depending on the input parameters, the number of maxima of the time function of the contact force can vary from one to three. It can be seen from the obtained results that in this case the graph has three local maxima.

Analysis of different results obtained during the simulation with different initial data indicates that the dependence of the contact force on the time under impact is a rather complex function that cannot be represented as simple functions such as, for example, the Heaviside function. In particular, analysis of dynamic loss of stability of a rod should include the possible forms of the function of the force applied to the end of the rod.

One of the advantages of the finite-element method over the semi-analytic method is that it is possible to determine the set of parameters at any time (for example, the values of displacements, deformations, stresses, etc.). This option makes it possible to focus closer on the wave processes evolving in the rod under impact, which ultimately determine the form of the contact force. In particular, Fig. 4 shows the time dependence of the longitudinal displacement of the points of the cross-section located half the rod length away from the front end of the rod.

It can be seen that a superposition of waves propagating in the rod occurs. The method also makes it possible to compare the behavior of different points of the rod with the results of natural experiments.

Comparison of the results obtained by two methods

Since these methods are based on different concepts and assumptions, it is particularly interesting to carry out comparative analysis of the simulation results. Fig. 2 shows such a comparison for the time dependences of the contact forces obtained by the semi-analytical and finite-element methods.

Comparing the obtained graphs, we can see the perfect qualitative agreement of their forms. Quantitative comparison of the calculated results reveals that the values of the contact time differ by 11 %, and the values of the local extrema of the functions by 9 %. Based on the simulation data, we can conclude that the results obtained by both methods are in good agreement, and that the contact force and contact time were determined correctly.

If we compare the obtained dependences in more detail, we can notice small oscillations

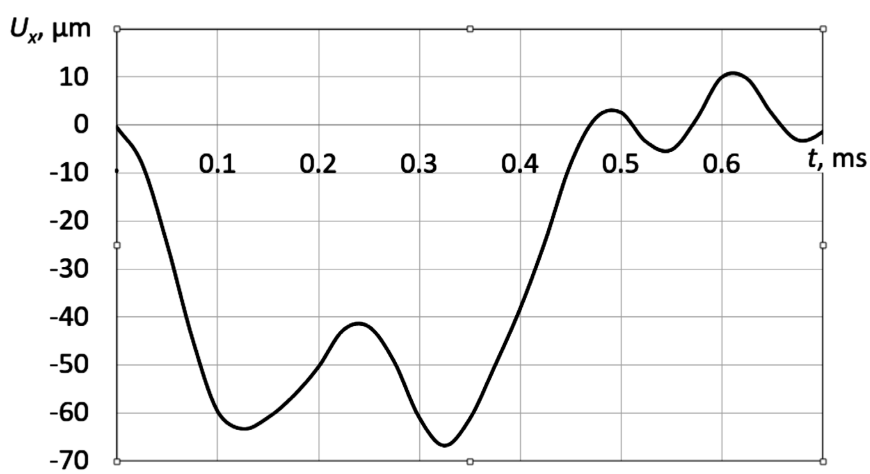

Fig. 4. Time dependence of the longitudinal displacement of the cross-section of the rod; the cross section is located half the rod length away from the front end of the rod (the result was obtained by finite element modeling)

on the curve obtained by the finite-element method, absent from the curve obtained by the semi-analytic method. These oscillations can be attributed to the influence of edge effects which the semi-analytic method does not take into account.

Comparison of the results of the natural experiment with the data of the two calculation methods

Natural experiments served to verify the semi-analytical and finite-element methods.

The natural experiment conducted at the National Taiwan University involved a cylinder with a radius of 10.0 mm and a length of 30.1 mm, made of a material whose parameters are given in Table 1.

One was the ends of the cylinder was fixed, and a steel impactor approached the second free end at the initial moment time at a speed of 0.3225 m/s (the impactor was ball-shaped). A piezoelectric film 28 μm in thickness and 7 × 3 mm in size, serving as a piezoelectric sensor, was attached to the free end of the rod. The principle of measuring the time of contact between the cylinder and the impactor was based on the piezoelectric effect; the time was measured depending on the mass of the impactor (Table 2).

A comparison of the results of finite-element simulation and semi-analytical calculation with the results of natural experiments is also given in Table 2.

It can be seen from the results given that the greatest error is observed for the finite-element method at intermediate values of the impactor mass. The error is minimal for the smallest and the largest of the selected masses.

A decrease in the error is observed for the semi-analytic method with increasing mass of the impactor. Thus, the minimal values of the impactor mass yield the greatest discrepancy between the results of the semi-analytic method and the results obtained in the natural experiment and by finite-element modeling. However, this discrepancy decreases with an increase in the mass of the impactor.

Conclusions

In this paper, we have used two fundamentally different methods (finite-element and semi-analytic) to study the dynamic process of an impact on a perfectly elastic rod in the longitudinal direction. In particular, the contact force and the interaction time have been determined. The wave processes occurring in the rod upon impact have also been studied.

The obtained results were in agreement, and it seemed logical to conduct a natural experiment to verify both methods. In addition, the approximation of the graphs makes it possible to use such functions to solve related problems. In particular, using approximated functions in studies on dynamic loss of rod stability can allow to correctly compare simulation with experimental results, since impact interaction is

Table 2

Comparison of the experimental results with the data obtained by the two calculation methods

Value of the impactor parameter		Time of the contact interaction, μs			Error of the method, %	
Diameter, m	Mass, g	Natural experiment	FEM	SEM	FEM	SEM
3.16	0.13	36.42	31.20	15.32	14.33	57.93
4.75	0.44	46.83	60.75	23.63	29.72	49.54
5.56	0.71	52.26	78.09	27.90	49.42	46.61
6.34	1.04	60.05	81.61	31.71	35.90	47.19
9.51	3.51	84.96	106.21	47.94	25.01	43.57
12.73	8.46	111.10	121.71	64.82	9.54	41.66

Abbreviations: FEM is the finite-element method, SEM is the semi-analytical method (mathematical modeling).

easier to carry out than, for example, step impacts. The latter are extremely popular in various model problems but using them in natural experiments is not possible yet.

The study was conducted within the framework of the Russian-Taiwanese grant of the Russian Foundation for Basic Research No. 16-51-52025 MHT_a.

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Received 16.12.2016, accepted 13.11.2017.

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